How to use direct comparison for integrals containing an asymptote

Hey all,

Just wanted to work out an example of direct comparison for improper integrals in which the integrand has a vertical asymptote, in full detail. Here’s our example:

Determine whether \( \int_0^3 \frac{dx}{\sqrt{4x^2 + 9x}} \) converges or diverges.

- First of all, it is not immediately clear how to evaluate the indefinite integral \( \int \frac{dx}{\sqrt{4x^2 + 9x}} \) symbolically. In fact, there is no "nice" antiderivative for the function \((4x^2 + 9x)^{-1/2}\), and besides, the question is only asking us to determine convergence/divergence, so we do not have to evaluate the integral.
- We cannot use asymptotic analysis. Why? Asymptotic analysis as we’ve learned it applies only to integrals where the upper bound of integration is \( \infty \). Asymptotic analysis uses facts about how quickly a function grows/decays as \( x \to \infty \). Here, the values of \( x \) in our integral are inside the interval \([0, 2]\) — infinity is too far away to be relevant, and so asymptotic analysis will not help us.
- Thus, our only remaining option is to use direct comparison. Here are some tips on how to proceed—there are other ways, and this method does not always work, but this is how I would approach this particular problem:
  - Remember to always ask why the integral is improper. In this case, the integral is improper because the denominator of the integrand is zero when \( x = 0 \).
  - Next, isolate the "problem" by factoring: The reason the denominator is zero at \( x = 0 \) is because the polynomial inside the radical contains \( x \) as a factor. That is, \( \frac{1}{\sqrt{4x^2 + 9x}} = \frac{1}{\sqrt{x(4x + 9)}} = \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{4x + 9}} \)
  - We’ve now factored the integrand into two parts: The “problem term” \( \frac{1}{\sqrt{x}} \) (which is the term that causes the integrand to become undefined at \( x = 0 \)) and the “bystander term” \( \frac{1}{\sqrt{4x + 9}} \), which is always defined (never infinity) when \( x \) is in the interval of integration \([0, 2]\).
  - The next step is to ask how small/big can the bystander term be, when \( x \) is limited to the interval of integration? In our case I claim that when \( 0 \leq x \leq 2 \), we have \( \frac{1}{5} \leq \frac{1}{\sqrt{4x + 9}} \leq \frac{1}{3} \). Step-by-step:
    - \( 0 \leq 4x + 9 \leq 25 \)
    - \( 3 \leq \sqrt{4x + 9} \leq 5 \)
    - \( \frac{1}{3} \leq \frac{1}{\sqrt{4x + 9}} \leq \frac{1}{5} \)
  - Note that when taking reciprocals you have to flip the inequalities: If \( a \) and \( b \) are both positive (or both negative) and \( a \leq b \), then \( \frac{1}{a} \geq \frac{1}{b} \).
  - Now, when \( 0 \leq x \leq 2 \), the problem term \( \frac{1}{\sqrt{x}} \) is always positive (its minimum value is \( \frac{1}{\sqrt{2}} \)). Thus, we can multiply that last inequality above by the problem term without flipping anything:
    - \( \frac{1}{\sqrt{x}} \leq \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{4x + 9}} \leq \frac{1}{3} \)
  - Finally, we can integrate the whole inequality from 0 to 2 without affecting the ordering:
    - \( \int_0^2 \frac{dx}{\sqrt{x}} \leq \int_0^2 \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{4x + 9}} dx \leq \int_0^2 \frac{dx}{\sqrt{3}} \)
  - The very last step is to determine the convergence/divergence behavior of the "easy" integrals above—the ones on either side of the integral we care about. You can either do this by evaluating them directly, or by appealing to the fact that \( \int_0^1 \frac{dx}{x^2} \) converges (so \( \int_0^2 \frac{dx}{x^2} \) converges too, as changing the upper endpoint to 2 does not introduce new problems—the constants 3 and 5 also do not affect the convergence behavior). In our case, the integrals on either side of our integral converge, so our integral converges as well.
  - As a bonus of direct comparison, we get bounds on our integral. Even though we cannot evaluate our integral, by evaluating the side integrals, we see that our integral converges to a value between \( \frac{\pi^2}{15} \approx 0.566 \) and \( \frac{\pi^2}{15} \approx 0.943 \). (Our integral’s actual value is \( \approx 0.839 \).)
  - This of course may seem like a long process but the general idea is: Isolate the problem term in the integral and, essentially, replace the "bystander term" with the constants that bound its value above and below (1/5 and 1/3 in our example). Then determine whether the resulting simple-looking integrals converge or diverge.

As with most calculus problems, practice makes perfect. Here are couple more example problems:

1. If we change our example above to \( \int_0^2 \frac{dx}{\sqrt{4x^2 + 9x^2}} \) the 9x term is now 9x^2, the integral now diverges. Can you figure out why?

2. Problem 7.7.24: \( \int_0^\pi \frac{2 - \sin \phi}{\phi^2} d\phi \). Here, the problem term is \( \frac{1}{\phi^2} \) and the bystander term is \( 2 - \sin \phi \).

3. Problem 7.7.20, which is on your homework, can also be approached this way. In this case, what is the problem term? What is the bystander term?

4. \( \int_2^3 \frac{dx}{\sqrt{x^2 - 4}} \). Here you have to factor the denominator in order to isolate the problem term properly.

5. \( \int_0^{\pi/3} \frac{\cos x}{x^{1/4}} \). Here bounding the bystander term is very slightly more difficult than usual—you can’t just get away with \(-1 \leq \cos x \leq 1\) this time.