CME 303/MATH 220: PROBLEM SET 2
DUE 9AM, FRIDAY, OCTOBER 12, 2018

Problem 1. Consider the PDE

$$u_t + uu_x = 0, \; u(x,0) = \phi(x).$$

Suppose that $\phi' \geq -C$, where $C > 0$. Show that the PDE has a C^1 solution on $\mathbb{R}_x \times [0, \frac{1}{C})_t$. Show also that for $t \in [0, \frac{1}{C})$, u_x satisfies the estimate

$$u_x(x,t) \geq \frac{1}{t - C^{-1}}.$$

(Note that the right hand side is negative!) (Hint: Consider the difference quotients $\frac{u(\xi_2(t),t)-u(\xi_1(t),t)}{\xi_2(t)-\xi_1(t)}$, where $x = \xi_j(t)$ are the projected characteristic curves emanating from the point x_j on the x axis.)

Problem 2. Show that the only solution $u \in \mathcal{D}'(\mathbb{R})$ of $u' = 0$ is $u = c$, c a constant function.

Hint: $u' = 0$ means that $u(\phi') = 0$ for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$. You need to show that there is a constant c such that $u(\psi) = \int c\psi \, dx$ for all $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$. To do so, consider when $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ is of the form $\psi = \phi'$, $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$, paying particular attention to the issue of compact supports. Then write an arbitrary ψ as a linear combination of a fixed $\phi_0 \in \mathcal{C}_c^\infty(\mathbb{R})$ and the derivative ϕ' of some $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$.

Problem 3. Consider the PDE

$$au_x + u_y = 0.$$

We already know that the C^1 solutions are of the form $u(x,y) = f(x - ay)$, $f \in C^1(\mathbb{R})$.

(1) Show that if f is merely piecewise continuous (or if you wish locally integrable), then the so defined u still solves the PDE in the sense of distributions.
(2) Suppose now that f is a distribution, e.g. $f = \delta_0$. Can you make sense of the formula $u(x,y) = f(x - ay)$? That is, find a procedure giving a distribution u when you start with $f \in \mathcal{D}'(\mathbb{R})$ which depends continuously on f (i.e. if $f_j \to f$ then the corresponding distributions satisfy $u_j \to u$) and make sure that if f is a piecewice continuous function, then you end up with the function $u(x,y) = f(x - ay)$.
(3) Check that the distribution you defined still solves the PDE.

Problem 4. In $t \geq 0$, consider the conservation law

$$u_t + f(u)_x = 0, \; u(x,0) = \phi(x).$$

Suppose u is C^1 except that it has a jump discontinuity along the C^1-curve given by $x = \xi(t)$, and $u|_{t=0} = \phi$. Let

$$\Omega_+ = \{(x,t) : \; x > \xi(t)\}, \; \Omega_- = \{(x,t) : \; x < \xi(t)\},$$

and let $u_\pm = u|_{\Omega_\pm}$, so $u_\pm$ are C^1 on $\overline{\Omega_\pm}$. Show that u is a weak solution of the PDE if and only if u_+ and u_- solve the PDE in the classical sense, i.e. the PDE holds pointwise in $\Omega_+ \cup \Omega_-$ and the initial condition holds there at $t = 0$, and

$$(u_-(\xi(t), t) - u_+(\xi(t), t))\xi'(t) = f(u_-(\xi(t), t)) - f(u_+(\xi(t), t)).$$

This is called the *Rankine-Hugoniot jump condition.*

Using this, in $t \geq 0$, find a weak solution of Burgers' equation

$$u_t + uu_x = 0, \; u(x, 0) = \phi(x)$$

with

$$\phi(x) = \begin{cases} 1, & x < 0, \\ 0, & x > 0, \end{cases}$$

which is constant to the left, resp. the right, of a curve $x = \xi(t)$. This is called a *shock wave.*