MATH 220: Problem Set 2
Solutions

Problem 1. As discussed in class and in the lecture notes, the classical solutions of Burgers' equations are constant along the projected characteristic curves, and the latter are given implicitly by $x = \phi(r)t + r$. In particular, the classical (C^1) solutions exist until two characteristic curves intersect. To see when this happens, consider two distinct characteristic curves given by $x = \xi_j(t)$, $j = 1, 2$, with $\xi_j(t) = \phi(r_j)t + r_j$, i.e. these are the curves through $(r_j, 0)$, and w.l.o.g assume that $r_1 < r_2$ (we can assume this as characteristic curves are distinct, and so $r_1 \neq r_2$). If these intersect at (x, t) then $\xi_1(t) = \xi_2(t)$, i.e.

$$(\phi(r_2) - \phi(r_1))t = r_1 - r_2.$$

By the mean value theorem, $\phi(r_2)-\phi(r_1) = (r_2-r_1)\phi'(r_0)$ for some $r_0 \in (r_1, r_2)$, so at the point of intersection $-\phi'(r_0)t = 1$ (as $r_1 \neq r_2$). So $\phi'(r_0) \neq 0$ and $t = \frac{1}{-\phi'(r_0)}$. As we want $t \geq 0$, we must have $-\phi'(r_0) > 0$ and as $-\phi' \leq C$, we deduce that $t \geq C^{-1}$, as claimed. So the solution exists on $\mathbb{R}_x \times [0, \frac{1}{C})_t$.

For the second part, consider $t \in [0, \frac{1}{C})$, and note that $t - C^{-1} < 0$, and so $(t - C^{-1})^{-1} < 0$ also. Next consider two distinct points $(x_j, t), j = 1, 2$, with $x_1 < x_2$. As the solution is C^1 here, there are unique characteristic curves passing through these points given as $\xi_j(t) = \phi(r_j)t + r_j$. As the characteristic curves do not intersect on $\mathbb{R}_x \times [0, \frac{1}{C})_t$, we immediately have $r_1 < r_2$ (as $x_1 < x_2$).

Then noting that $x_j = \xi_j(t)$, we have

$$\frac{u(\xi_2(t), t) - u(\xi_1(t), t)}{\xi_2(t) - \xi_1(t)} = \frac{\phi(r_2) - \phi(r_1)}{(\phi(r_2) - \phi(r_1))t + r_2 - r_1} = \frac{1}{t + \frac{r_2 - r_1}{\phi(r_2) - \phi(r_1)}}$$

if $\phi(r_2) \neq \phi(r_1)$, and 0 otherwise. But by the mean value theorem as above, $\phi(r_2) - \phi(r_1) = (r_2 - r_1)\phi'(r_0)$ for some $r_0 \in (r_1, r_2)$, with $\phi'(r_0) \neq 0$, and so

$$\frac{u(x_2) - u(x_1)}{x_2 - x_1} = \begin{cases} 0 & \text{if } \phi(r_2) = \phi(r_1) \\ \frac{1}{t + \frac{1}{\phi'(r_0)}} & \text{if } \phi(r_2) \neq \phi(r_1) \end{cases}.$$

Now suppose $\phi(r_2) > \phi(r_1)$. Then $\phi'(r_0) > 0$ (as we already assumed $r_2 > r_1$), and so $\left(t + \frac{1}{\phi'(r_0)}\right) > 0$, as $t \in [0, \frac{1}{C})$, which implies $\left(t + \frac{1}{\phi'(r_0)}\right)^{-1} > 0$. On the other hand, if $\phi(r_2) < \phi(r_1)$, then $-C \leq \phi'(r_0) < 0$, and so $\frac{1}{\phi'(r_0)} \leq -C^{-1}$, which finally implies that

$$t + \frac{1}{\phi'(r_0)} \leq t - C^{-1} < 0,$$

Thus, whether $\phi(r_2) > \phi(r_1)$, $\phi(r_2) = \phi(r_1)$ or $\phi(r_2) < \phi(r_1)$, we deduce that

$$\frac{u(x_2) - u(x_1)}{x_2 - x_1} \geq \frac{1}{t - C^{-1}},$$

for all $x_1, x_2 \in \mathbb{R}_x$, and $t \in [0, \frac{1}{C})$.

But

$$\frac{\partial u}{\partial x}(\xi_1(t), t) = \lim_{r_2 \to r_1} \frac{u(\xi_2(t), t) - u(\xi_1(t), t)}{\xi_2(t) - \xi_1(t)},$$

and so $\frac{\partial u}{\partial x}(\xi_1(t), t) \geq \frac{1}{t - C^{-1}}$ as claimed.

Problem 2. $u' = 0$ in $\mathcal{D}'(\mathbb{R})$ means that $u(\phi') = 0$, $\forall \phi \in \mathcal{C}_c^\infty(\mathbb{R})$. And we want to show that:

$$\exists c \in \mathbb{R},\ \forall \psi \in \mathcal{C}_c^\infty(\mathbb{R}),\ u(\psi) = \int_{\mathbb{R}} c\psi(x)dx.$$

We will need a little lemma (only the converse) proved below.

Lemma 1. *If $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$, then $\phi' \in \mathcal{C}_c^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} \phi'(x)dx = 0$. Conversely, if $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ and $\int_{\mathbb{R}} \phi(x)dx = 0$, then $\exists\, \psi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\phi = \psi'$.*

Proof. For the forward direction, let $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$, and so there exists $R > 0$ such that $\text{supp}(\phi) \subset (-R, R)$. This implies that ϕ' is C^∞ (by definition) and $\phi' = 0$ on $\mathbb{R} \setminus (-R, R)$, and so $\phi' \in \mathcal{C}_c^\infty(\mathbb{R})$. Finally

$$\int_{\mathbb{R}} \phi'(x)dx = \int_{-R}^{R} \phi'(x)dx = \phi(R) - \phi(-R) = 0.$$

For the converse, assume that $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$, with $\text{supp}(\phi) \subset (-R, R)$, and $\int_{\mathbb{R}} \phi(x)dx = 0$. Then if we define $\psi(x) = \int_{-\infty}^{x} \phi(t)dt$, we have that $\psi = 0$ on $\mathbb{R} \setminus (-R, R)$, and ψ is C^∞ as $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$. Differentiating ψ we also get that $\psi' = \phi$. $\square$

Let's choose a $\phi_0 \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \phi_0(x)dx \neq 0$ (recall the construction of bump functions in class that are non-negative), and w.l.o.g assume that $\int_{\mathbb{R}} \phi_0(x)dx = 1$. Then for an arbitrary $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$, we can write $\psi = a_\psi \phi_0 + (\psi - a_\psi \phi_0)$, with $a_\psi \in \mathbb{R}$. Now if we choose $a_\psi = \int_{\mathbb{R}} \psi(x)dx$, then letting $\xi_\psi = \psi - a_\psi \phi_0$, and noticing that $\xi_\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ (as $\mathcal{C}_c^\infty(\mathbb{R})$ is a vector space), we have

$$\int_{\mathbb{R}} \xi_\psi(x)dx = \int_{\mathbb{R}} \psi(x)dx - a_\psi \int_{\mathbb{R}} \phi_0(x)dx = \int_{\mathbb{R}} \psi(x)dx - a_\psi = 0,$$

and so by Lemma 1, there exists $\phi_\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\xi_\psi = \phi'_\psi$.

Therefore

$$\begin{aligned} u(\psi) &= u(a_\psi \phi_0 + \phi'_\psi) \\ &= a_\psi u(\phi_0) + u(\phi'_\psi) \quad \text{(linearity)} \\ &= a_\psi u(\phi_0) \quad \text{(assumption on } u) \\ &= u(\phi_0) \int_{\mathbb{R}} \psi(x) dx \quad \text{(choice of } a_\psi) \\ &= c \int_{\mathbb{R}} \psi(x) dx, \end{aligned}$$

with $c = u(\phi_0)$, which is indeed a constant. The only remaining detail is to show that c does not depend on the choice of ϕ_0. To that extent, suppose that $\tilde{\phi_0} \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \tilde{\phi_0}(x)dx = 1$, but $\phi_0 \neq \tilde{\phi_0}$. Then $\int_{\mathbb{R}}(\phi - \tilde{\phi_0})(x)dx = 0$, and by Lemma 1 there exists $\hat{\phi} \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\hat{\phi}' = \phi - \tilde{\phi_0}$. Then $u(\phi_0) - u(\tilde{\phi_0}) = u(\phi_0 - \tilde{\phi_0}) = u(\hat{\phi}') = 0$, which implies u is independent of the choice of ϕ_0.

Problem 3. We consider here the PDE (transport equation):

$$au_x + u_y = 0. \tag{1}$$

(1) Suppose $f \in L^1_{loc}(\mathbb{R})$. We are asked to verify that $u(x,y) = f(x - ay)$ is a solution of the PDE (1) in the sense of distributions. Since $f \in L^1_{loc}(\mathbb{R})$, we can write for any $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$

$$\begin{aligned} (au_x + u_y)(\phi) &= -u(a\phi_x + \phi_y) \qquad \text{(distributional derivative)} \\ &= -\int_{\mathbb{R}^2} f(x - ay)\,(a\phi_x + \phi_y)\,(x,y)dx\,dy, \\ &= -\int_{\mathbb{R}^2} f(\xi)\phi_\eta(\xi,\eta)d\xi\,d\eta, \\ &= -\int_{\mathbb{R}} f(\xi)\left(\int_{\mathbb{R}} \phi_\eta(\xi,\eta)d\eta\right)\,d\xi \\ &= 0 \;(\phi \in \mathcal{C}_c^\infty(\mathbb{R}^2)) \end{aligned}$$

where we did the change of variables $\xi = x - ay$ and $\eta = y$, and used the Fubini theorem to separate the integrals (can be applied here since $f \in L^1_{loc}$). The last step follows because $\int_{\mathbb{R}} \phi_\eta(\xi,\eta)d\eta = 0$, which is a consequence of Lemma 1 from Problem 2.

(2) We proceed in the same fashion as we did in class to handle the cases of derivative and multiplication. Meaning that we are going to start with f merely piecewise continuous (or locally integrable) and see how it works, then infer what the distribution is when f is a distribution.

First, to emphasize the analogy, let us define the rule sending $f \in L^1_{loc}(\mathbb{R})$ to $u \in L^1_{loc}(\mathbb{R}^2)$ as a map $\tau_a : f \mapsto u$:

$$(\tau_a f)(x,y) = f(x - ay).$$

Then we need to express the distribution $\iota_{\tau_a f}$ associated to $\tau_a f$ in terms of ι_f, i.e. f should only enter on the right hand side via ι_f. So for any test function $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, we have

$$\begin{aligned}\iota_{\tau_a f}(\psi) &= \int_{\mathbb{R}^2} (\tau_a f)(x,y)\psi(x,y)dx\,dy \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x-ay)\psi(x,y)dx \right) dy.\end{aligned}$$

Again, by the change of variables $\xi = x - ay,\ \eta = y$, we get

$$\begin{aligned}\iota_{\tau_a f}(\psi) &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(\xi)\psi(\xi + a\eta, \eta)dx \right) dy \\ &= \int_{\mathbb{R}} f(\xi) \left(\int_{\mathbb{R}} \psi(\xi + a\eta, \eta)dx \right) dy \\ &= \iota_f(\Phi_{-a}\psi),\end{aligned}$$

where $\Phi_{-a}\psi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ is the test function given by

$$(\Phi_{-a}\psi)(\xi) = \int_{\mathbb{R}} \psi(\xi + a\eta, \eta)d\eta.$$

So now we make the definition that if f is replaced by a distribution, i.e. $f \in \mathcal{D}'(\mathbb{R})$, then $\tau_a f \in \mathcal{D}'(\mathbb{R}^2)$ is defined such that for any test function $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$,

$$(\tau_a f)(\psi) = f(\Phi_{-a}\psi), \text{ where } (\Phi_a\psi)(\xi) = \int_{\mathbb{R}} \psi(\xi - a\eta, \eta)d\eta. \tag{2}$$

(3) By definition, $au_x + u_y$ is the distribution such that, for all $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$, $(au_x + u_y)(\psi) = -u(a\psi_x + \psi_y)$. So we just need to prove that this vanishes for all ψ when $u = \tau_a f$. By definition of $\tau_a f$ in 2, we have

$$u(a\psi_x + \psi_y) = f(\Phi_{-a}(a\psi_x + \psi_y)),$$

and moreover by the definition of Φ_{-a}, we have

$$\Phi_{-a}(a\psi_x + \psi_y) = \int_{\mathbb{R}} (a\psi_x + \psi_y)(\xi + a\eta, \eta)d\eta.$$

Now observe that if we define $\phi(\xi,\eta) = \psi(\xi + a\eta, \eta)$, then $\phi \in \mathcal{C}_c^\infty(\mathbb{R})$ and

$$\phi_\eta(\xi,\eta) = a\psi_x(\xi + a\eta) + \psi_y(\xi + a\eta, \eta),$$

by the chain rule, and therefore by another application of Lemma 1 of Problem 2, we obtain

$$\begin{aligned}\Phi_{-a}(a\psi_x + \psi_y) &= \int_{\mathbb{R}} (a\psi_x + \psi_y)(\xi + a\eta, \eta)d\eta \\ &= \int_{\mathbb{R}} \phi_\eta(\xi,\eta)d\eta \\ &= 0.\end{aligned}$$

Therefore, eventually, for all $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$,

$$\begin{aligned}(au_x + u_y)(\psi) &= -u(a\psi_x + \psi_y) \\ &= -f(\Phi_{-a}(a\psi_x + \psi_y)) \\ &= -f(0) \\ &= 0,\end{aligned}$$

which exactly means that $au_x + u_y = 0$ in the sense of distributions.

Problem 4. In $t \geq 0$, we consider the conservation law

$$u_t + f(u)_x = 0, \ u(x,0) = \phi(x), \tag{3}$$

and suppose that u is C^1 except that it has a jump discontinuity along a C^1-curve given by $x = \xi(t)$.
Let us show that u is a weak solution of the PDE (3) if and only if u_+ and u_- solve the PDE in the classical sense.
First, let's suppose that u is a weak solution. Then, for all $\psi \in \mathcal{C}_c^\infty(\mathbb{R} \times [0, +\infty))$

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} (\ u\psi_t + f(u)\psi_x)\, dx\ dt + \int_{-\infty}^{+\infty} \phi(x)\psi(0,x)dx = 0. \tag{4}$$

Now let choose ψ such that $\psi(x,0) = 0$, and break up the first integral into the regions Ω_+ and Ω_-. We then get

$$\begin{aligned}0 &= \int_0^{+\infty} \int_{-\infty}^{+\infty} (\ u\psi_t + f(u)\psi_x)\, dx\ dt \\ &= \int\int_{\Omega_-} (\ u_-\psi_t + f(u_-)\psi_x)\, dx\ dt + \int\int_{\Omega_+} (\ u_+\psi_t + f(u_+)\psi_x)\, dx\ dt\end{aligned} \tag{5}$$

Now since $u_\pm$ is C^1 on $\Omega_\pm$, we can use the classical divergence theorem and the fact that ψ has compact support and $\psi(x,0) = 0$ to write

$$\begin{aligned}\int\int_{\Omega_\pm} (\ u_\pm\psi_t + f(u_\pm)\psi_x)\, dx\ dt &= -\int\int_{\Omega_\pm} ((\ u_\pm)_t + (f(u_\pm))_x)\, \psi dx\ dt \\ &\quad \pm \int_{x=\xi(t)} (\ u_\pm\psi\nu_2 + f(u_\pm)\psi\nu_1)\, ds,\end{aligned} \tag{6}$$

where $\nu = (\nu_1, \nu_2)$ is the outward unit normal to Ω_- (so also the opposite of the outward normal to Ω_+).

Since u is a weak solution, and u is smooth on either side of the curve $x = \xi(t)$, $u_\pm$ is a strong solution of (3) in $\Omega_\pm$ and

$$\int\int_{\Omega_\pm} ((\ u_\pm)_t + (f(u_\pm))_x)\, \psi dx\ dt = 0. \tag{7}$$

Therefore, for all smooth function ψ, we have

$$\int_{x=\xi(t)} (\, u_-\psi\nu_2 + f(u_-)\psi\nu_1)\, ds - \int_{x=\xi(t)} (\, u_+\psi\nu_2 + f(u_+)\psi\nu_1)\, ds = 0, \qquad (8)$$

which leads to

$$u_-(\xi(t),t)\nu_2 + f(u_-(\xi(t),t))\nu_1 = u_+(\xi(t),t)\nu_2 + f(u_+(\xi(t),t))\nu_1, \qquad (9)$$

and

$$\begin{aligned} f(u_-(\xi(t),t)) - f(u_+(\xi(t),t)) &= -\frac{\nu_2}{\nu_1}(u_-(\xi(t),t) - u_+(\xi(t),t)) \\ &= \xi'(t)(u_-(\xi(t),t) - u_+(\xi(t),t)), \end{aligned} \qquad (10)$$

which is exactly the *Rankine-Hugoniot jump condition*.
Hence u_+ and u_- solve the PDE in the classical sense.

Conversely, let's suppose that u_+ and u_- solve the PDE in the classical sense. Then (7) and (10) hold. By performing the previous operations in reverse order and using similar arguments, we get that u is indeed a weak solution.

Now let's look at the Burgers' equation with particular initial condition:

$$\begin{cases} u_t + uu_x = 0,\ x \in \mathbb{R},\ t > 0, \\ u(x,0) = \phi(x) = \begin{cases} 1,\ x < 0 \\ 0,\ x > 0 \end{cases} \end{cases} \qquad (11)$$

and let's find a weak solution using the previous analysis.
From the method of characteristics for first-order quasi-linear PDEs, we know that we have an implicit solution where u is smooth, namely $u = \phi(x - ut)$. Moreover u is constant along the projected characteristic curves given by $x_r(t) = \phi(r)t + r$.

- If $r < 0$, then $\phi(r) = 1$, which implies that the characteristic curves are

  $$x_r(t) = t + r,\ r < 0, \qquad (12)$$

  and the solution $u = u_-$ should equal 1 along those curves,

- If $r > 0$, then $\phi(r) = 0$, which implies that the characteristic curves are

  $$x_r(t) = r,\ r > 0, \qquad (13)$$

  and the solution $u = u_+$ should equal 0 along those curves.

Furthermore, at the jump, we have the *Rankine-Hugoniot jump condition*:

$$\frac{(u_-)^2}{2} - \frac{(u_+)^2}{2} = \xi'(t)(u_- - u_+), \qquad (14)$$

which reduces to

$$\xi'(t) = \frac{1}{2}. \tag{15}$$

Moreover, the curve $x = \xi(t)$ contains the point $(x,t) = (0,0)$ (we indeed have a discontinuity for the initial function at $x = 0$). Therefore the curve of discontinuity is given by $x = \dfrac{t}{2}$ and the weak solution is:

$$u(x,t) = \begin{cases} 1, \ x < \dfrac{t}{2} \\ 0, \ x > \dfrac{t}{2} \end{cases}, \tag{16}$$

which means that the discontinuity (shock) is moving at speed $1/2$.