MATH 220: PROBLEM SET 4
DUE 9AM, FRIDAY, OCTOBER 26, 2018

Problem 1.

(i) Find the general C^2 solution of the PDE

$$u_{xx} - u_{xt} - 6u_{tt} = 0$$

by reducing it to a system of first order PDEs.

(ii) Show that if $f, g \in \mathcal{D}'(\mathbb{R})$, and we define new distributions $v, w \in \mathcal{D}'(\mathbb{R}^2)$ as in Problem 3 of Problem Set 2, i.e. formally $v(x,t) = f(3x+t)$, $w(x,t) = g(-2x+t)$, then $u = v + w$ solves the PDE in (1). (Hint: use the result of Problem 3 of Problem Set 2, and factor our second order operator. This should only take a few lines.)

Problem 2. Solve the wave equation on the line:

$$u_{tt} - c^2 u_{xx} = 0,\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x),$$

with

$$\phi(x) = \begin{cases} 0, & x < -1, \\ 1+x, & -1 < x < 0, \\ 1-x, & 0 < x < 1, \\ 0, & x > 1. \end{cases}$$

and

$$\psi(x) = \begin{cases} 0, & x < -1, \\ 2, & -1 < x < 1, \\ 0, & x > 1. \end{cases}$$

Also describe in $t > 0$ where the solution vanishes, and where it is C^∞, and compare it with the general results discussed in lecture (Huygens' principle and propagation of singularities).

Problem 3. Consider the PDE

$$u_{tt} - \nabla \cdot (c^2 \nabla u) + qu = 0,\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x), \tag{1}$$

where $c, q \geq 0$, depend on x only, and c is bounded between positive constants, i.e. for some $c_1, c_2 > 0$, $c_1 \leq c(x) \leq c_2$ for all $x \in \mathbb{R}^n$. Assume that u is C^2 throughout this problem, and u is real-valued. (All calculations would go through if one wrote $|u_t|^2$, etc., in the complex valued case.)

(i) Fix $x_0 \in \mathbb{R}^n$ and $R_0 > 0$, and for $t < \frac{R_0}{c_2}$, let

$$E(t) = \int_{|x-x_0|<R_0-c_2t} (u_t^2 + c(x)^2|\nabla u|^2 + q(x)u^2)\,dx.$$

Show that E is decreasing with t (i.e. non-increasing). (Hint: to make sure you don't forget anything in the calculation, do it first on the line, when $n = 1$.)

(ii) Suppose that $\operatorname{supp}\phi, \operatorname{supp}\psi \subset \{|x| \leq R\}$, i.e. are 0 outside this ball. Show that $u(x,t) = 0$ if $t \geq 0$, $|x| > R + c_2 t$, i.e. the wave indeed propagates at speed $\leq c_2$.

(iii) Show that there is at most one real-valued C^2 solution of (1).

Problem 4. Consider the wave equation on $\mathbb{R}^n$:

$$u_{tt} - c^2\Delta u = f,\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x),$$

and write $x = (x', x_n)$ where $x' = (x_1, \dots, x_{n-1})$

(i) Show that if

$$f(x', x_n, t) = f(x', -x_n, t),\ \phi(x', x_n) = \phi(x', -x_n),\ \psi(x', x_n) = \psi(x', -x_n)$$

for all x and t, i.e. if f, ϕ, ψ are all even functions of x_n, then u is an even function of x_n as well. (Hint: Consider $u(x', x_n, t) - u(x', -x_n, t)$, show that it solves the homogeneous wave equation with 0 initial conditions.)

(ii) Show that if

$$f(x', x_n, t) = -f(x', -x_n, t),\ \phi(x', -x_n) = -\phi(x', -x_n),\ \psi(x', x_n) = -\psi(x', -x_n)$$

for all x and t, i.e. if f, ϕ, ψ are all odd functions of x_n, then u is an odd function of x_n as well.

(iii) If u is continuous, and is an odd function of x_n, show that $u(x', 0, t) = 0$ for all x' and t.

(iv) If u is a C^1 and is an even function of x_n, show that $\partial_{x_n} u(x', 0, t) = 0$ for all x' and t.

These facts will enable us to solve the wave equation in the half space $x_n > 0$ with Dirichlet or Neumann boundary conditions later in the course.

Problem 5. Use the maximum principle for Laplace's equation on $\mathbb{R}^n$ to show the following statement: Suppose that $u \in C^2(\mathbb{R}^n)$ and $\Delta u = 0$. Suppose moreover that $u(x) \to 0$ at infinity uniformly in the following sense:

$$\sup_{|x|>R} |u(x)| \to 0$$

as $R \to \infty$. Then $u(x) = 0$ for all $x \in \mathbb{R}^n$. (Hint: Apply the maximum principle shown in class for the ball $\Omega = \{x :\ |x| < R\}$ and for both u and $-u$.)

Use this to show that the solution of Laplace's equation on $\mathbb{R}^n$:

$$\Delta u = f,$$

with f given, is unique in the class of functions u such that $u \in C^2(\mathbb{R}^n)$ and $u(x) \to 0$ at infinity uniformly.