MATH 220: Problem Set 4
Solutions

Problem 1.

(1) Let's find the general C^2 solution of the PDE

$$u_{xx} - u_{xt} - 6u_{tt} = 0, \tag{1}$$

by reducing it to a system of first order PDEs (by the way, this is an elliptic PDE).
We are looking for a, b, c, d that formally verify:

$$\begin{aligned} \partial_{xx} - \partial_{xt} - 6\partial_{tt} &= (a\partial_x + b\partial_t)(c\partial_x + d\partial_t) \\ &= ac\partial_{xx} + (ad + bc)\partial_{xt} + bd\partial_{tt}. \end{aligned} \tag{2}$$

So we get the (under-determined) system:

$$\begin{cases} ac = 1 \\ ad + bc = -1 \\ bd = -6. \end{cases} \tag{3}$$

From the first equation, let us simply take $a = c = 1$. Then the system reduces to

$$\begin{cases} d + b = -1 \\ bd = -6, \end{cases} \tag{4}$$

which gives $b = 2$ and $d = -3$.

Therefore we can write $u_{xx} - u_{xt} - 6u_{tt} = 0$ as $(\partial_x + 2\partial_t)(\partial_x - 3\partial_t)u = 0$. Now let $v = (\partial_x - 3\partial_t)u$. Then v verifies $v_x + 2v_t = 0$ (first order linear PDE!). And we know that the solution writes $v(x,t) = h(t - 2x)$ for some $h \in \mathcal{C}^1$.
Now for u we have the system:

$$u_x - 3u_t = h(t - 2x). \tag{5}$$

Using the method of characteristics, we get the following equations:

$$\begin{cases} x_r'(s) = 1, & x_r(0) = 0, \\ t_r'(s) = -3, & t_r(0) = r, \\ v_r'(s) = h(t_r(s) - 2x_r(s)), & v_r(0) = \phi(r), \end{cases} \tag{6}$$

for some function $\phi \in \mathcal{C}^2$.
Therefore we have $x_r(s) = s$, $t_r(s) = -3s + r$, and

$$v_r'(s) = h(-3s + r - 2s) = h(-5s + r), \tag{7}$$

and by integrating from $s = 0$, we get:

$$\begin{aligned} v_r(s) &= \int_0^s h(-5s' + r)ds' + \phi(r) \\ &= -\frac{1}{5}\int_r^{-5s+r} h(y)dy + \phi(r), \end{aligned} \tag{8}$$

after a change of variables. Now, since we have $s = x$ and $r = t + 3x$, we finally get:

$$\begin{aligned} u(x,t) &= \frac{1}{5}\int_{t-2x}^{t+3x} h(y)dy + \phi(t + 3x) \\ &= f(t + 3x) + g(t - 2x), \end{aligned} \tag{9}$$

for some $f, g \in \mathcal{C}^2$.

Reciprocally, we verify that u of the form $u(x,t) = f(t + 3x) + g(t - 2x)$ for $f, g \in \mathcal{C}^2$ indeed solves the PDE.

(2) For an arbitrary $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ we have to show that

$$u(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) = v(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) + w(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) = 0. \tag{10}$$

But from **Problem 3** (PSET 2), we have:

$$v(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) = v((\partial_x - 3\partial_t)(\phi_x + 2\phi_t)) = 0, \tag{11}$$

and similarly,

$$w(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) = w((\partial_x + 2\partial_t)(\phi_x - 3\phi_t)) = 0. \tag{12}$$

Problem 2. Let's solve the wave equation on the line:

$$u_{tt} - c^2 u_{xx} = 0, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \tag{13}$$

with

$$\phi(x) = \begin{cases} 0, & x < -1, \\ 1 + x, & -1 < x < 0, \\ 1 - x, & 0 < x < 1, \\ 0, & x > 1, \end{cases} \tag{14}$$

and

$$\psi(x) = \begin{cases} 0, & x < -1, \\ 2, & -1 < x < 1, \\ 0, & x > 1. \end{cases} \tag{15}$$

We know from the course (method of characteristics) that the solution is:

$$u(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy. \tag{16}$$

Now, notice first that since $c, t \geq 0$, then $x - ct \leq x + ct$. There are 10 cases to consider for the solution:

1. If $x + ct < -1$ and $x - ct < -1$ (domain 1), then $u(x,t) = 0$.
2. If $-1 \leq x + ct < 0$ and $x - ct < -1$ (domain 2), then
$$u(x,t) = \frac{1}{2}(1+x+ct+0) + \frac{1}{2c}\int_{-1}^{x+ct} 2dy = \left(\frac{1}{2} + \frac{1}{c}\right)(1+x+ct). \tag{17}$$
3. If $0 \leq x + ct < 1$ and $x - ct < -1$ (domain 3), then
$$u(x,t) = \frac{1}{2}(1-x-ct+0) + \frac{1}{2c}\int_{-1}^{x+ct} 2dy = \frac{1}{2}(1-x-ct) + \frac{1}{c}(1+x+ct). \tag{18}$$
4. If $1 \leq x + ct$ and $x - ct < -1$ (domain 4), then
$$u(x,t) = 0 + \frac{1}{2c}\int_{-1}^{-1} 2dy = \frac{2}{c}. \tag{19}$$
5. If $-1 \leq x + ct < 0$ and $-1 < x - ct < 0$ (domain 5), then
$$u(x,t) = \frac{1}{2}(1+x+ct+1+x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} 2dy = 1 + x + 2t. \tag{20}$$
6. If $0 \leq x + ct < 1$ and $-1 < x - ct < 0$ (domain 6), then
$$u(x,t) = \frac{1}{2}(1-x-ct+1+x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} 2dy = 1 - ct + 2t. \tag{21}$$
7. If $1 \leq x + ct$ and $-1 \leq x - ct < 0$ (domain 7), then
$$u(x,t) = \frac{1}{2}(0+1+x-ct) + \frac{1}{2c}\int_{x-ct}^{1} 2dy = \frac{1}{2}(1+x-ct) + \frac{1}{c}(1-x+ct). \tag{22}$$

8. If $0 \leq x+ct < 1$ and $0 \leq x-ct < 1$ (domain 8), then

$$u(x,t) = \frac{1}{2}(1-x-ct+1-x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} 2dy = 1-x+2t. \quad (23)$$

9. If $1 \leq x+ct$ and $0 \leq x-ct < 1$ (domain 9), then

$$u(x,t) = \frac{1}{2}(1-x-ct) + \frac{1}{2c}\int_{x-ct}^{1} 2dy = \frac{1}{2}(1+x-ct) + \frac{1}{c}(1-x+ct). \quad (24)$$

10. If $1 \leq x+ct$ and $1 \leq x-ct$ (domain 10), then $u(x,t) = 0$.

Finally, as we can now see, $u(x,t)$ vanishes in region (1) and (10). Since $\phi(x) = 0$ and $\psi(x) = 0$ in $|x| > 1$, this result corresponds to Huygens' principle.
Moreover, $u(x,t)$ is $\mathcal{C}^1$ except on the lines $x+ct = -1$, $x+ct = 0$, $x+ct = 1$, $x-ct = -1$, $x-ct = 0$ and $x-ct = 1$. Since $\phi(x)$ and $\psi(x)$ are $\mathcal{C}^1$ everywhere except at $x = -1, 0, 1$, this result corresponds to the propagation of singularities: $u(x,t)$ is $\mathcal{C}^1$ near (x,t) if ϕ and ψ are such near $x \pm ct$.

Problem 3. Let $f(x,t) = u_t^2 + c(x)^2\left|\nabla u\right|^2 + q(x)u^2$, so that we can simply write $E(t) = \displaystyle\int_{|x-x_0|<R_0-c_2t} f(x,t)dx$.

(i) To investigate the variation of $E(t)$, we would like to derive E with respect to t. The difficulty here is that t appears in the very domain of integration, so that we have an integral with parameter.
As the hint suggests, let us first look at the $1D$ case. Then

$$\begin{aligned} E(t) &= \int_{|x-x_0|<R_0-c_2t} f(x,t)dx, \\ &= \int_{x_0-R_0+c_2t}^{x_0+R_0-c_2t} f(x,t)dx. \end{aligned} \quad (25)$$

And at this point, we can use the following result (exercise):

Lemma 1. *Suppose $a(t)$ and $b(t)$ are $\mathcal{C}^1$ functions, and $g(x,t)$ continuous in x and $\mathcal{C}^1$ in t. Then if $G(t) = \displaystyle\int_{a(t)}^{b(t)} g(x,t)dx$, then*

$$G'(t) = b'(t)f(b(t),t) - a'(t)f(a(t),t) + \int_{a(t)}^{b(t)} \frac{\partial g}{\partial t}(x,t)dx. \quad (26)$$

Now by applying it in our problem, we get:

$$E'(t) = -c_2 f(x_0 - R_0 + c_2 t, t) - c_2 f(x_0 + R_0 - c_2 t, t) + \int_{x_0 - R_0 + c_2 t}^{x_0 + R_0 - c_2 t} \frac{\partial f}{\partial t}(x,t)dx. \tag{27}$$

Therefore we see that when the moving domain is in $1D$, it is rather simple to manage the derivative of $E(t)$. Now let us go back to the general case. If we look at the moving domain in our case, we have that $\{|x - x_0| < R_0 - c_2 t\} = B(x_0, R_0 - c_2 t)$, the ball centered at x_0 and of radius $R_0 - c_2 t$. Therefore we have radial symmetry which naturally inclines us to you the polar change of coordinates to get:

$$\begin{aligned} E(t) &= \int_{|x-x_0|<R_0-c_2t} f(x,t)dx, \\ &= \int_{r=0}^{r=R_0-c_2t} \left(\int_{S^{n-1}} f(x_0 + r\omega, t) r^{n-1} d\omega \right) dr. \end{aligned} \tag{28}$$

where $S^{n-1} = \{x \in \mathbb{R}^n,\ |x| = 1\}$ is the unit sphere.

Then applying the lemma on $E(t)$, we get:

$$\begin{aligned} E'(t) &= -c_2 \int_{S^{n-1}} f(x_0 + (R_0 - c_2 t)\omega, t)(R_0 - c_2 t)^{n-1} d\omega \\ &+ \int_{r=0}^{r=R_0-c_2t} \left(\int_{S^{n-1}} \frac{\partial f}{\partial t}(x_0 + r\omega, t) r^{n-1} d\omega \right) dr \\ &= -c_2 \int_{|x-x_0|=R_0-c_2t} f(x,t)dx + \int_{|x-x_0|<R_0-c_2t} \frac{\partial f}{\partial t}(x,t)dx, \end{aligned} \tag{29}$$

where we used again change of variables/coordinates.

Now we can write, replacing $f(x,t)$ by its value:

$$\begin{aligned} E'(t) &= -c_2 \int_{|y-x_0|=R_0-c_2t} \left(u_t^2 + c(y)^2 \left|\nabla u\right|^2 + q(y) u^2 \right) dy \\ &+2 \int_{|x-x_0|<R_0-c_2t} \left(u_t u_{tt} + c(x)^2 \nabla u \cdot \nabla u_t + q(x) u u_t \right) dx, \end{aligned} \tag{30}$$

And here we see that from $c(x)^2 \nabla u \cdot \nabla u_t$, we would like to get back u_t and $\nabla \cdot \big(c(x)^2 \nabla u\big)$, which invites us to use the divergence theorem to get:

$$\begin{aligned} \int_{|x-x_0|<R_0-c_2t} c(x)^2 \nabla u \cdot \nabla u_t dx &= \int_{|y-x_0|=R_0-c_2t} c(y)^2 \frac{\partial u}{\partial \nu} u_t dy \\ &- \int_{|x-x_0|<R_0-c_2t} \nabla \cdot \big(c(x)^2 \nabla u\big) u_t dx, \end{aligned} \tag{31}$$

so that:

$$
\begin{aligned}
E'(t) &= \int_{|y-x_0|=R_0-c_2t} \left(2c(y)^2 \frac{\partial u}{\partial \nu} u_t - c_2 \left(u_t^2 + c(y)^2 \left|\nabla u\right|^2 + q(y)u^2\right)\right) dy \\
&\quad +2 \int_{|x-x_0|<R_0-c_2t} u_t \left(u_{tt} - \nabla \cdot \left(c(x)^2 \nabla u\right) + q(x)u\right) dx \\
&= \int_{|y-x_0|=R_0-c_2t} \left(2c(y)^2 \frac{\partial u}{\partial \nu} u_t - c_2 \left(u_t^2 + c(y)^2 \left|\nabla u\right|^2 + q(y)u^2\right)\right) dy,
\end{aligned} \tag{32}
$$

since u verifies the PDE. Now, using the same "trick" as in class, we have that:

$$
\begin{aligned}
\int_{|y-x_0|=R_0-c_2t} 2c(y)^2 \frac{\partial u}{\partial \nu} u_t dy &\leq \int_{|y-x_0|=R_0-c_2t} c(y) 2c(y) \left|\frac{\partial u}{\partial \nu}\right| |u_t| \, dy \\
&\leq c_2 \int_{|y-x_0|=R_0-c_2t} 2c(y) \left|\nabla u\right| |u_t| \, dy \\
&\leq c_2 \int_{|y-x_0|=R_0-c_2t} \left(c(y)^2 \left|\nabla u\right|^2 + u_t^2\right) dy.
\end{aligned} \tag{33}
$$

because $c(x) \geq 0$ and $c(x) < c_2$. And finally, putting everything together, we get that:

$$
\begin{aligned}
E'(t) &\leq -c_2 \int_{|y-x_0|=R_0-c_2t} q(y)u^2 dy \\
&\leq 0,
\end{aligned} \tag{34}
$$

and E is indeed non-increasing with t.

(ii) Now let's suppose that suppϕ, supp$\psi \subset \{|x| \leq R\} = B(0, R)$. Let us consider (x_0, t_0) such that $t_0 \geq 0$ and $|x_0| > R + c_2 t_0$. The aim here is to prove that $u(x_0, t_0) = 0$.

So if you take $R_0 = c_2 t_0$ and consider, for $t < \frac{R_0}{c_2} = t_0$

$$
E(t) = \int_{|x-x_0|<c_2(t_0-t)} \left(u_t^2 + c(x)^2 \left|\nabla u\right|^2 + q(x)u^2\right) dx, \tag{35}
$$

then from (i), E is non-increasing. Furthermore, since suppϕ, supp$\psi \subset \{|x| \leq R\}$ while $\{|x - x_0| < c_2 t_0\} \subset \{|x| > R\}$, we can assert that

$$
E(0) = \int_{|x-x_0|<c_2t_0} \left(\psi^2 + c(x)^2 \left|\nabla \phi\right|^2 + q(x)\phi^2\right) dx = 0. \tag{36}
$$

Moreover, from the expression of $E(t)$, E is non-negative, and therefore $E(t) = 0$, $0 \leq t < t_0$.
Consequently, $u_t = 0$, $\nabla u = 0$ as $c(x) \geq c_1 > 0$ and $q(x) \geq 0$. This implies that u is constant in the cone $\{|x - x_0| \leq c_2(t_0 - t)\}$. Finally, noticing that $u(x_0, 0) = \phi(x_0) = 0$ since $x_0 > R$, we conclude that the constant is zero and $u(x_0, t_0) = 0$. Therefore the wave indeed propagates at finite speed $\leq c_2$.

(iii) Let's suppose that we have u_1 and u_2 solutions of the PDE. Then by linearity, w defined as $w = u_2 - u_1$ solves the following PDE:

$$\begin{cases} w_{tt} - \nabla \cdot \big(c(x)^2 \nabla w\big) + q(x)w = 0, \\ w(x,0) = 0, \\ w_t(x,0) = 0. \end{cases} \tag{37}$$

Hence, from part (ii), we directly get that for any (x_0, t_0), $w(x_0, t_0) = 0$. Therefore u_1 and u_2 are identical and there is at most one real-value $\mathcal{C}^2$ solution of the PDE.

Problem 4. Consider the wave equation on $\mathbb{R}^n$:

$$\begin{cases} u_{tt} - c^2 \Delta u = f, \\ u(x,0) = \phi(x), \\ u_t(x,0) = \psi(x), \end{cases} \tag{38}$$

and we write $x = (x', x_n)$ with $x' = (x_1, x_2, ..., x_{n-1})$.

(i) We suppose here that f, ϕ and ψ are all even functions of x_n. Following the hint, let's consider $v(x', x_n, t) = u(x', x_n, t) - u(x', -x_n, t)$ and let's find a PDE for v.
Using the assumption on the data, we get:

- $v_{tt}(x', x_n, t) = u_{tt}(x', x_n, t) - u_{tt}(x', -x_n, t)$,
- $\Delta v(x', x_n, t) = \Delta u(x', x_n, t) - \Delta u(x', -x_n, t)$,
- $f(x', x_n, t) - f(x', -x_n, t) = 0$,
- $v(x', x_n, 0) = \phi(x', x_n) - \phi(x', -x_n) = 0$,
- $v_t(x', x_n, 0) = \psi(x', x_n) - \psi(x', -x_n) = 0$.

Therefore v verifies the following PDE:

$$\begin{cases} v_{tt} - c^2 \Delta v = 0, \\ v(x,0) = 0, \\ v_t(x,0) = 0, \end{cases} \tag{39}$$

and by uniqueness of solution, this implies that v is identically zero, which gives

$$u(x', x_n, t) = u(x', -x_n, t), \tag{40}$$

meaning the solution u conserves the symmetry of the data.

(ii) We suppose here that f, ϕ and ψ are all odd functions of x_n. Let's consider $w(x', x_n, t) = u(x', x_n, t) + u(x', -x_n, t)$ and let's find a PDE for w.
Using the assumption on the data, we get:

- $w_{tt}(x', x_n, t) = u_{tt}(x', x_n, t) + u_{tt}(x', -x_n, t)$,
- $\Delta w(x', x_n, t) = \Delta u(x', x_n, t) + \Delta u(x', -x_n, t)$,
- $f(x', x_n, t) + f(x', -x_n, t) = 0$,
- $v(x', x_n, 0) = \phi(x', x_n) + \phi(x', -x_n) = 0$,
- $v_t(x', x_n, 0) = \psi(x', x_n) + \psi(x', -x_n) = 0$.

Therefore w verifies the following PDE:

$$\begin{cases} w_{tt} - c^2 \Delta w = 0, \\ w(x, 0) = 0, \\ w_t(x, 0) = 0, \end{cases} \tag{41}$$

and by uniqueness of solution, this implies that w is identically zero, which gives

$$u(x', x_n, t) = -u(x', -x_n, t), \tag{42}$$

meaning the solution u conserves the symmetry of the data.

(iii) From (ii), we have that $u(x', 0, t) = -u(x', -0, t)$ and by continuity of u, $u(x', 0, t) = 0$.

(iv) From (i), we have that $u(x', x_n, t) = u(x', -x_n, t)$ and since u is $\mathcal{C}^1$, we get that $\partial_{x_n} u(x', x_n, t) = -\partial_{x_n} u(x', -x_n, t)$. Now taking $x_n = 0$, we conclude that $\partial_{x_n} u(x', 0, t) = -\partial_{x_n} u(x', 0, t)$ and $\partial_{x_n} u(x', 0, t) = 0$.

Problem 5. We consider $u \in \mathcal{C}^2(\mathbb{R}^n)$, such that $\Delta u = 0$ and $\sup_{|x|>R} |u(x)| \to 0$ as $R \to +\infty$ (uniformly vanishing at infinity).

Without loss of generality, it suffices to prove that $u(0) = 0$. This is because for any arbitrary point $x \in \mathbb{R}^n$, we can simply translate the coordinates, which preserves the PDE and the decay conditions continue to hold (you should check this). Now proving that $u(0) = 0$ is equivalent to proving that for any $\epsilon > 0$, $|u(0)| \leq \epsilon$.

So fix $\epsilon > 0$. By the statement of the problem (the decay conditions on u), there exists some $R > 0$ such that $\sup_{|x|>R} |u(x)| < \epsilon$. Then by continuity of u (or a density argument), the estimate will also hold on the set $\overline{\{|x| > R\}} = \{|x| \geq R\}$, and so $\sup_{|x|\geq R} |u(x)| < \epsilon$. Now we can apply the maximum principle for

Laplace's equation to the domain $\{|x| < R\}$ (which is bounded, so we can apply the theorem!), and conclude that

$$|u(0)| \leq \sup_{|x|<R} |u(x)| \leq \sup_{|x|=R} |u(x)| \leq \sup_{|x|\geq R} |u(x)| < \epsilon.$$

Suppose u and v are $\mathcal{C}^2(\mathbb{R}^n)$ and uniformly vanishing at infinity. Suppose furthermore that $\Delta u = f$, $\Delta v = f$ with f given. Consider $w = u - v$. Then w has the same properties as u and v (you should check that w indeed satisfies the decay estimates), and satisfies $\Delta w = 0$. But from the first part of the problem, this implies that $w = 0$, which means $u = v$. Therefore the solution of the Laplace equation on $\mathbb{R}^n$ for that class of functions is unique (if it exists).