CME 303/MATH 220: PROBLEM SET 5
DUE 10AM WEDNESDAY, OCTOBER 31, 2018

Problem 1. Suppose that f is (piecewise) continuous on $\mathbb{R}^n$ with $|x|^N f(x)$ bounded for some $N > n$ (or indeed simply that $f \in L^1(\mathbb{R}^n)$). Throughout this problem, $a \in \mathbb{R}^n$.

    (i) Let $f_a(x) = f(x-a)$. Show that $(\mathcal{F}f_a)(\xi) = e^{-ia\cdot\xi}(\mathcal{F}f)(\xi)$.

    (ii) Let $g_a(x) = e^{ix\cdot a}f(x)$. Show that $(\mathcal{F}g_a)(\xi) = (\mathcal{F}f)(\xi - a)$.

    (iii) Show that $(\mathcal{F}^{-1}f_a)(x) = e^{ia\cdot x}(\mathcal{F}^{-1}f)(x)$.

    (iv) Show that $(\mathcal{F}^{-1}g_a)(x) = (\mathcal{F}^{-1}f)(x+a)$.

Problem 2. Use part (i) of Problem 1 to show that $(\mathcal{F}(\partial_j f))(\xi) = i\xi_j(\mathcal{F}f)(\xi)$ if f is C^1 and $|x|^N f$, $|x|^N \partial_j f$ are bounded for some $N > n$.

Problem 3. Find the Fourier transform on $\mathbb{R}$ of the following functions:

    (i) $f(x) = H(a - |x|)$, where $a > 0$ and H is the Heaviside step function, so $H(t) = 1$ if $t > 0$, $H(t) = 0$ if $t < 0$.

    (ii) $f(x) = H(x)e^{-ax}$, where $a > 0$.

    (iii) $f(x) = |x|^n e^{-a|x|}$, where $a > 0$, and $n \geq 0$ integer.

    (iv) $f(x) = (1 + x^2)^{-1}$. (Hint: use that if $f = \mathcal{F}^{-1}g$ then $g = \mathcal{F}f$ by the Fourier inversion formula. Rewrite $(1 + x^2)^{-1}$ as partial fractions (factor the denominator).)

Problem 4. Find the Fourier transform on $\mathbb{R}^3$ of the function $f(x) = |x|^n e^{-a|x|}$, where $a > 0$ and $n \geq -1$ integer. (Hint: express the integral in the Fourier transform in spherical coordinates.)

Problem 5. Let $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k$, and write $\mathbb{R}^n \ni x = (y, z) \in \mathbb{R}^m \times \mathbb{R}^k$. Suppose that $f \in C^1(\mathbb{R}^m \times \mathbb{R}^k)$ and $|z|^K f$, $|z|^K \partial_{x_j} f$ are bounded for all $j = 1, \ldots, n$, and $K > k$. Define the *partial Fourier transform* of f by

$$(\mathcal{F}_z f)(y, \zeta) = \int_{\mathbb{R}^k} e^{-iz\cdot\zeta} f(y, z)\, dz, \ y \in \mathbb{R}^m, \ \zeta \in \mathbb{R}^k.$$

Show that

    (i) $(\mathcal{F}_z D_{z_j} f)(y, \zeta) = \zeta_j (\mathcal{F}_z f)(y, \zeta)$.

    (ii) $(\mathcal{F}_z D_{y_j} f)(y, \zeta) = (D_{y_j}(\mathcal{F}_z f))(y, \zeta)$.

Note: Under appropriate additional assumptions, as for the full Fourier transform, the formulae $\mathcal{F}_z(z_j f) = -D_{\zeta_j}\mathcal{F}_z f$, $\mathcal{F}_z(y_j f) = y_j\mathcal{F}_z f$, and the analogous formulae for

$$(\mathcal{F}_\zeta^{-1}\psi)(y, z) = (2\pi)^{-k}\int_{\mathbb{R}^k} e^{iz\cdot\zeta}\psi(y, \zeta)\, dz$$

also hold, but you do not need to prove this (but you should know these!).