Problem 1. Suppose that $f$ is (piecewise) continuous on $\mathbb{R}^n$ with $|x|^N f(x)$ bounded for some $N > n$ (or indeed simply that $f \in L^1(\mathbb{R}^n)$). Throughout this problem, $a \in \mathbb{R}^n$.

(i) Let $f_a(x) = f(x-a)$. Show that $(\mathcal{F}f_a)(\xi) = e^{-ia \cdot \xi} (\mathcal{F}f)(\xi)$.

(ii) Let $g_a(x) = e^{ia \cdot x} f(x)$. Show that $(\mathcal{F}g_a)(\xi) = (\mathcal{F}f)(\xi - a)$.

(iii) Show that $(\mathcal{F}^{-1}g_a)(x) = e^{ia \cdot x} (\mathcal{F}^{-1}f)(x)$.

(iv) Show that $(\mathcal{F}^{-1}g_a)(x) = (\mathcal{F}^{-1}f)(x + a)$.

Problem 2. Use part (i) of Problem 1 to show that $(\mathcal{F}(\partial_t f))(\xi) = i\xi_t (\mathcal{F}f)(\xi)$ if $f$ is $C^1$ and $|x|^N f$, $|x|^N \partial_t f$ are bounded for some $N > n$.

Problem 3. Find the Fourier transform on $\mathbb{R}$ of the following functions:

(i) $f(x) = H(a - |x|)$, where $a > 0$ and $H$ is the Heaviside step function, so $H(t) = 1$ if $t > 0$, $H(t) = 0$ if $t < 0$.

(ii) $f(x) = H(x) e^{-ax}$, where $a > 0$.

(iii) $f(x) = |x|^n e^{-a|x|}$, where $a > 0$, and $n \geq 0$ integer.

(iv) $f(x) = (1 + x^2)^{-1}$. (Hint: use that if $f = \mathcal{F}^{-1} g$ then $g = \mathcal{F} f$ by the Fourier inversion formula. Rewrite $(1 + x^2)^{-1}$ as partial fractions (factor the denominator).)

Problem 4. Find the Fourier transform on $\mathbb{R}^3$ of the function $f(x) = |x|^n e^{-a|x|}$, where $a > 0$ and $n \geq -1$ integer. (Hint: express the integral in the Fourier transform in spherical coordinates.)

Problem 5. Let $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k$, and write $\mathbb{R}^n \ni x = (y, z) \in \mathbb{R}^m \times \mathbb{R}^k$. Suppose that $f \in C^1(\mathbb{R}^m \times \mathbb{R}^k)$ and $|z|^K f$, $|z|^K \partial_j f$ are bounded for all $j = 1, \ldots, n$, and $K > k$. Define the partial Fourier transform of $f$ by

$$(\mathcal{F}_z f)(y, \zeta) = \int_{\mathbb{R}^k} e^{-ix \cdot \zeta} f(y, z) \, dz, \quad y \in \mathbb{R}^m, \quad \zeta \in \mathbb{R}^k.$$  

Show that

(i) $(\mathcal{F}_z D_z f)(y, \zeta) = \zeta_j (\mathcal{F}_z f)(y, \zeta)$.

(ii) $(\mathcal{F}_z D_y f)(y, \zeta) = (D_y (\mathcal{F}_z f))(y, \zeta)$.

Note: Under appropriate additional assumptions, as for the full Fourier transform, the formulae $\mathcal{F}_z (z_j f) = -D_{z_j} \mathcal{F}_z f$, $\mathcal{F}_z (y_j f) = y_j \mathcal{F}_z f$, and the analogous formulae for

$$(\mathcal{F}_z^{-1} \psi)(y, z) = (2\pi)^{-k} \int_{\mathbb{R}^k} e^{iz \cdot \zeta} \psi(y, \zeta) \, dz$$

also hold, but you do not need to prove this (but you should know these!).