MATH 220: Problem Set 5
Solutions

Problem 1. Let $f \in L^1(\mathbb{R}^n)$ and $a \in \mathbb{R}^n$. The whole problem is a matter of change of variables with integrals.

(i)

$$\begin{aligned}(\mathcal{F}f_a)(\xi) &= \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f_a(x)dx = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x-a)dx \\ &= \int_{\mathbb{R}^n} e^{-i(a+y)\cdot\xi} f(y)dy = e^{-ia\cdot\xi}\int_{\mathbb{R}^n} e^{-iy\cdot\xi} f(y)dy \\ &= e^{-ia\cdot\xi}(\mathcal{F}f_a)(\xi).\end{aligned} \tag{1}$$

(ii)

$$\begin{aligned}(\mathcal{F}g_a)(\xi) &= \int_{\mathbb{R}^n} e^{-ix\cdot\xi} g_a(x)dx = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} e^{ia\cdot x} f(x)dx \\ &= \int_{\mathbb{R}^n} e^{-i(\xi-a)\cdot x} f(x)dx \\ &= (\mathcal{F}f)(\xi-a).\end{aligned} \tag{2}$$

(iii)

$$\begin{aligned}\left(\mathcal{F}^{-1}f_a\right)(\xi) &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f_a(x)dx = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} f(x-a)dx \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i(a+y)\cdot\xi} f(y)dy = e^{ia\cdot\xi}(2\pi)^{-n}\int_{\mathbb{R}^n} e^{iy\cdot\xi} f(y)dy \\ &= e^{ia\cdot\xi}\left(\mathcal{F}^{-1}f_a\right)(\xi).\end{aligned} \tag{3}$$

(iv)

$$\begin{aligned}\left(\mathcal{F}^{-1}g_a\right)(\xi) &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} g_a(x)dx = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{ia\cdot x} f(x)dx \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i(\xi+a)\cdot x} f(x)dx \\ &= \left(\mathcal{F}^{-1}f\right)(\xi+a).\end{aligned} \tag{4}$$

Problem 2. Let $f \in \mathcal{C}^1(\mathbb{R}^n)$, and $|x|^N f(x)$, $|x|^N \partial_j f(x)$ bounded with $N > n$.

Let's write $x = (x_1, ..., x_j, ..., x_n)$ and take $h_j = (0, ..., h, ..., 0) = he_j$ (zero everywhere except for the j^{th} coordinate).
We have

$$\left(\mathcal{F}(\partial_j f)\right)(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}(\partial_j f)(x)dx = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \lim_{h\to 0} \frac{f(x+h_j)-f(x)}{h} dx \tag{5}$$

If we can switch limit and integral, we are done. Therefore we need to justify the inversion.
Let's take $h_j^k = (0, ..., h^k, ..., 0) = h^k e_j$, with $(h^k)_{k\geq 0}$ such that $h^k \to 0$ as $k \to +\infty$. Now we can consider $g_k(x) = \dfrac{f(x+h_j^k)-f(x)}{h^k}$.

- g_k is integrable for all $k \geq 0$ since f continuous and $|x|^N f(x)$ is bounded,
- $g_k(x) \to \partial_j f(x)$ as $k \to +\infty$ pointwise,
- for all $k \geq 0$, $|g_k(x)| \leq \phi(x)$, with $\phi \in L^1(\mathbb{R}^n)$, $\phi(x) \geq 0$. Indeed, by the mean value theorem, there exists $\eta_j^k \in (x_j^k, x_j^k + h^k)$ such that $g_k(x) = \partial_j f(x_1, ..., \eta_j^k, ..., x_n)$. Now, for instance, for $|x| < 1$, there exists $M_1 > 0$ such that $|\partial_j f(x)| \leq M_1$ ($\partial_j f$ continuous), and for $|x| > 1$, there exists $M_2 > 0$ such that $|\partial_j f(x)| \leq \dfrac{M_1}{|x|^N}$ by assumption.

  Hence, $|g_k(x)| \leq M_1 \chi_{\{|x|<1\}} + \dfrac{M_1}{|x|^N}\chi_{\{|x|>1\}} = \phi(x)$, with χ the indicator function.

Therefore, we can use the Lebesgue Dominated Convergence Theorem (LDCT) on $h_k(x;\xi) = e^{-ix\cdot\xi} g_k(x)$ (same conclusions since $|e^{-ix\cdot\xi}| = 1$) and write:

$$\begin{aligned}
\left(\mathcal{F}(\partial_j f)\right)(\xi) &= \int_{\mathbb{R}^n} \lim_{k\to+\infty} h_k(x;\xi)dx \\
&= \lim_{k\to+\infty} \int_{\mathbb{R}^n} h_k(x;\xi)dx \\
&= \lim_{k\to+\infty} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \frac{f(x+h_j^k)-f(x)}{h^k} dx \\
&= \lim_{k\to+\infty} \frac{e^{ih_j^k\cdot\xi}\left(\mathcal{F}f\right)(\xi) - \left(\mathcal{F}f\right)(\xi)}{h^k} \\
&= \left(\lim_{k\to+\infty} \frac{e^{ih_j^k\cdot\xi}-1}{h^k}\right)\left(\mathcal{F}f\right)(\xi) \\
&= \left(\lim_{k\to+\infty} \frac{e^{ih^k\xi_j}-1}{h^k}\right)\left(\mathcal{F}f\right)(\xi) \\
&= i\xi_j \left(\mathcal{F}f\right)(\xi)
\end{aligned} \tag{6}$$

In short, if f is $\mathcal{C}^1$ and $|x|^N f$, $|x|^N \partial_j f$ are bounded, with $N > n$, we can invert integral and limit. Also, we could have used integration by parts to get to the result.

An alternative of the LDCT is to use the following argumentation.
$(1+|x|)^{N'} \dfrac{f(x+h_j)-f(x)}{h}$ converges uniformly on the real line corresponding to the x_j-axis to $(1+|x|)^{N'} \partial_j f(x)$ for $N' < N$. Indeed, the difference quotient is $\displaystyle\int_0^1 \partial_j f(x+sh_j) ds$ by Taylor's theorem (or simply the fundamental theorem of calculus with a change of variable), and by assumption this integral is bounded by $C(1+|x|)^{-N}$. So first $(1+|x|^N)\dfrac{f(x+h_j)-f(x)}{h}$ is uniformly bounded (in $|h| \leq 1$, $h \neq 0$ and $x \in \mathbb{R}^n$).
Further, it differs from $\partial_j f(x)$ by $\displaystyle\int_0^1 (\partial_j f(x+sh) - \partial_j f(x)) ds$. Since $\partial_j f$ is continuous, it is uniformly continuous on compact sets, and since $(1+|x|)^N \partial_j f$ is bounded, $(1+|x|)^{N'} \partial_j f$ is uniformly continuous for $N' < N$. (The extra decay factor $(1+|x|)^{(N'-N)}$ makes sure it goes to 0 at infinity, which is why uniform continuity is easy.) Thus, given $\epsilon > 0$, for sufficiently small h, and for $s \in [0,1]$, one has for all $x \in \mathbb{R}^n$,

$$(1+|x|)^{N'} |\partial_j f(x+sh_j) - \partial_j f(x)| < \epsilon \tag{7}$$

Thus, $\left|(1+|x|)^{N'} \dfrac{f(x+h_j)-f(x)}{h} - (1+|x|)^{N'} \partial_j f(x)\right| < \epsilon$ for sufficiently small h, and so, with $(1+|x|)^{-N'}$ being integrable,

$$\int \left| \frac{f(x+h_j)-f(x)}{h} - \partial_j f(x) \right| dx < C'\epsilon, \tag{8}$$

and thus $\left| \displaystyle\int \frac{f(x+h_j)-f(x)}{h} dx - \int \partial_j f(x) dx \right| < C'\epsilon$ for small h. This gives the desired convergence (since you can add the extra multiplicative coefficient $e^{-ix\cdot\xi}$ with no significant effect on the conclusions).

Problem 3.

(i) $H(a-|x|) = 1$ if $-a < x < a$, and 0 otherwise. Therefore

$$(\mathcal{F}f)(\xi) \quad = \int_{\mathbb{R}} e^{-ix\xi} H(a-|x|) dx = \int_{-a}^{a} e^{-ix\xi} dx = \frac{e^{ia\xi} - e^{-ia\xi}}{i\xi}. \tag{9}$$

(ii) $H(x)e^{-ax}$ is 0 if $x < 0$. Therefore:

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{R}} e^{-ix\xi} e^{-ax} H(x) dx = \int_0^{+\infty} e^{-(i\xi+a)x} dx = \frac{1}{i\xi + a}. \tag{10}$$

(iii)

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{R}} e^{-ix\xi} |x|^n e^{-a|x|} dx = \int_0^{+\infty} x^n e^{-(i\xi+a)x} dx + \int_{-\infty}^0 (-x)^n e^{-(i\xi-a)x} dx \tag{11}$$

If we consider the first integral, doing integrations by part n times, we get:

$$\begin{aligned} \int_0^{+\infty} x^n e^{-(i\xi+a)x} dx &= (-1)^n \frac{n!}{(-a-i\xi)^n} \int_0^{+\infty} e^{-(i\xi-a)x} dx \\ &= (-1)^{n+1} \frac{n!}{(-a-i\xi)^{n+1}} = \frac{n!}{(a+i\xi)^{n+1}} \end{aligned} \tag{12}$$

Similarly, for the second term, we have

$$\int_{-\infty}^0 (-x)^n e^{-(i\xi-a)x} dx = \frac{n!}{(a-i\xi)^{n+1}} \tag{13}$$

Therefore

$$(\mathcal{F}f)(\xi) = \frac{n!}{(a+i\xi)^{n+1}} + \frac{n!}{(a-i\xi)^{n+1}}. \tag{14}$$

(iv) We can write

$$\frac{1}{1+x^2} = \frac{1}{2}\left(\frac{1}{1+ix} + \frac{1}{1-ix}\right). \tag{15}$$

Furthermore, we have

$$\begin{aligned} \left(\mathcal{F}^{-1} e^{-|x|}\right)(\xi) &= \frac{1}{2\pi}\left(\int_{-\infty}^0 e^{ix\xi} e^x dx + \int_0^{+\infty} e^{ix\xi} e^{-x} dx\right) \\ &= \frac{1}{2\pi}\left(-\int_{+\infty}^0 e^{-iy\xi} e^{-y} dy - \int_0^{-\infty} e^{-iy\xi} e^y dy\right) \\ &= \frac{1}{2\pi}\left(\int_0^{+\infty} e^{-iy\xi} e^{-y} dy + \int_{-\infty}^0 e^{-iy\xi} e^y dy\right) \\ &= \frac{1}{2\pi}\left(\mathcal{F}e^{-|x|}\right)(\xi) \\ &= \frac{1}{2\pi}\left(\frac{1}{1+i\xi} + \frac{1}{1-i\xi}\right), \end{aligned} \tag{16}$$

by change of variable and using the result from part (iii) with $n = 0$.
Now using the Fourier inversion formula, we deduce that:

$$\begin{aligned}\left(\mathcal{F}\frac{1}{1+x^2}\right)(\xi) &= \left(\mathcal{F}\frac{1}{2}\left(\frac{1}{1+ix}+\frac{1}{1-ix}\right)\right)(\xi)\\ &= \pi\left(\mathcal{F}\mathcal{F}^{-1}e^{-|x|}\right)(\xi)\\ &= \pi e^{-|\xi|}.\end{aligned} \tag{17}$$

Problem 4. We have that

$$\left(\mathcal{F}f\right)(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi}|x|^n e^{-a|x|}dx. \tag{18}$$

Let T be a linear transformation such that we rotate the z-axis to ξ (we fix $\xi \in \mathbb{R}^3$). Therefore $T([0,0,1]) = \xi$. Let $T([1,0,0]) = \xi_1$ and $T([0,1,0]) = \xi_2$. We can use spherical coordinates such that $x = (x_1, x_2, x_3)$ into (r, θ, ϕ), where $r = |x|$, θ is the angle between ξ and x, and ϕ is an angle of rotation about ξ from ξ_2.
Since $x \cdot \xi = |x||\xi|\cos(\theta)$, we obtain that

$$\begin{aligned}\left(\mathcal{F}f\right)(\xi) &= \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=+\infty}\int_{\phi=0}^{\phi=2\pi} e^{-ir|\xi|\cos(\theta)}r^n e^{-ar}r^2\sin(\theta)d\phi dr d\theta\\ &= 2\pi\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=+\infty} e^{-ir|\xi|\cos(\theta)}r^{n+2}e^{-ar}\sin(\theta)dr d\theta\\ &= 2\pi\int_0^{\pi}\frac{(n+2)!}{(a+i|\xi|\cos(\theta))^{n+3}}\sin(\theta)d\theta \text{ (from Pb3 (ii))}\\ &= 2\pi\frac{(n+1)!}{-i|\xi|}\left(\frac{1}{(a+i|\xi|)^{n+2}}-\frac{1}{(a-i|\xi|)^{n+2}}\right)\\ &= 2\pi i\frac{(n+1)!}{|\xi|}\left(\frac{1}{(a+i|\xi|)^{n+2}}-\frac{1}{(a-i|\xi|)^{n+2}}\right)\end{aligned} \tag{19}$$

Problem 5.

(i) By integration by part, we get

$$\begin{aligned}(\mathcal{F}_z D_{z_j} f)(y,\zeta) &= \frac{1}{i}\int_{\mathbb{R}^k} e^{-iz\cdot\zeta}\partial_{z_j}f(y,z)dz\\ &= -\frac{1}{i}\int_{\mathbb{R}^k}\partial_{z_j}\left(e^{-iz\cdot\zeta}\right)f(y,z)dz\\ &= \zeta_j\int_{\mathbb{R}^k} e^{-iz\cdot\zeta}f(y,z)dz\\ &= \zeta_j(\mathcal{F}_z f)(y,\zeta)\end{aligned} \tag{20}$$

(ii) By definition, we have that

$$(\mathcal{F}_z D_{y_j} f)(y,\zeta) \quad = \frac{1}{i}\int_{\mathbb{R}^k} e^{-iz\cdot\zeta}\partial_{y_j} f(y,z)dz \tag{21}$$

Now since f is C^1, and $|z|^K f$, $|z|^K \partial_{x_j} f$ are bounded with $K > k$, using the same argumentation as in Pb2, we can switch the derivative and the integral sign to finally obtain

$$(\mathcal{F}_z D_{y_j} f)(y,\zeta) \quad = D_{y_j}\int_{\mathbb{R}^k} e^{-iz\cdot\zeta} f(y,z)dz = (D_{y_j}\mathcal{F}_z f)(y,\zeta). \tag{22}$$