Problem 1.
(i) On $\mathbb{R}^3$, find the Fourier transform of the function $g(x) = |x|^{-1}$. (Hint: to do this efficiently, consider $g(x)$ as the limit of $g_a(x) = e^{-a|x|} |x|^{-1}$, and use your result from the last problem set.)

(ii) Solve $\Delta u = f$ on $\mathbb{R}^3$, where $f \in \mathcal{S}(\mathbb{R}^3)$, writing your answer as a convolution.

Problem 2. Find the Fourier transform on $\mathbb{R}^3$ of the distribution $u = \delta_{|x|-R}$, i.e. for $\psi \in \mathcal{S}(\mathbb{R}^3)$,

$$u(\psi) = R^2 \int_{S^2} \psi(R\omega) dS(\omega),$$

or in spherical coordinates

$$u(\psi) = R^2 \int_0^\pi \int_0^{2\pi} \psi(R\cos \theta, R\sin \theta \cos \phi, R\sin \theta \sin \phi) \sin \theta d\phi d\theta.$$

(Hint: use that $u$ is compactly supported, so you can evaluate the Fourier transform directly by applying it to $e^{-ix \cdot \xi}$, and use spherical coordinates centered around $\xi$.)

Problem 3. Write the solution of the wave equation on $\mathbb{R}^3 \times \mathbb{R}$,

$$u_{tt} = c^2 \Delta_x u, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x),$$

in a form that does not involve the Fourier transform by using convolutions.

Hint: Use the previous problem to deal with the $\psi$-term. To deal with the $\phi$ term, note that

$$\cos(c|\xi|t) = \frac{\partial}{\partial t} \left( \frac{\sin(c|\xi|t)}{c|\xi|} \right).$$

Problem 4. Show that if $u \in \mathcal{S}'(\mathbb{R}^n)$ then there is an integer $m \geq 0$ and $C > 0$ such that for all $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$|u(\phi)| \leq C \|\phi\|_m$$

where

$$\|\phi\|_m = \sum_{|\alpha| \leq m, |\beta| \leq m} \sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta \phi|.$$

Hints: This relies on the continuity of $u$ as a map $u : \mathcal{S} \to \mathbb{C}$. So suppose for the sake of contradiction that no such $m$ and $C$ exist; in particular for an integer $j > 0$, $m = j$ and $C = j$ do not work, i.e. there exists $\phi_j \in \mathcal{S}$ such that

$$|u(\phi_j)| > j\|\phi_j\|_j.$$

Note that $\phi_j$ cannot be 0 (for then $u(\phi_j)$ would vanish by linearity). Let $\psi_j = \frac{1}{j\|\phi_j\|_j} \phi_j$, so $\psi_j \in \mathcal{S}$, $\|\psi_j\|_j = \frac{1}{j}$ and

$$|u(\psi_j)| > j\|\psi_j\|_j = 1.$$

Now show that $\psi_j \to 0$ in $\mathcal{S}$ as $j \to \infty$, and use this to get a contradiction with the continuity of $u$. 

1
Problem 5. Suppose that \( u \in \mathcal{S}'(\mathbb{R}^n) \) and \( u \) has compact support. (This means that there is a function \( f \in C_c^\infty(\mathbb{R}^n) \) such that \( u = fu \); namely one would take \( f \) identically 1 on a neighborhood of the support of \( u \).) Show that the \( C^\infty \) function \( Fu \) satisfies
\[
|(Fu)(\xi)| \leq C(1 + |\xi|)^m
\]
for some \( C \) and \( m \).

Hint: Use the result of the previous problem and that \((Fu)(\xi) = u(\phi_\xi), \phi_\xi(x) = f(x)e^{-ix\cdot\xi}\).

Problem 6. Suppose that \( u, v \in \mathcal{S}'(\mathbb{R}^n) \) and \( v \) has compact support. Give a definition of \( u * v \in \mathcal{S}'(\mathbb{R}^n) \) that is consistent with the definition if one of the two distributions is in \( \mathcal{S}(\mathbb{R}^n) \).

Problem 7. Suppose that \( P = \sum_{|\alpha| \leq m} a_\alpha D^\alpha \) on \( \mathbb{R}^n \). A fundamental solution for \( P \) is a distribution \( E \in \mathcal{D}'(\mathbb{R}^n) \) such that \( PE = \delta_0 \). \( E \) is also called a Green’s function with pole at 0.

(i) Show that if \( E \) is a fundamental solution for \( P \) then for \( f \in C_c^\infty(\mathbb{R}^n) \), \( u = E * f \) solves \( Pu = f \).

(ii) Show that the same holds even if \( f \in C_c^0(\mathbb{R}^n) \) (or indeed \( f \in \mathcal{D}'(\mathbb{R}^n) \) with compact support).

You may assume that \( E \in \mathcal{S}'(\mathbb{R}^n) \) to use your results from the previous problem set, if you wish (though this is not strictly necessary).

(iii) Show that the distribution \( E \) given by the function \( \frac{1}{4\pi|x|} \) is a fundamental solution for \( \Delta \) in \( \mathbb{R}^3 \) in two different ways: using the Fourier transform, and directly.

Hint: For the direct calculation, to find \( \Delta E \), recall that \( \Delta E(\phi) = \Delta(\phi) \), and write the right hand side as \( -\lim_{\epsilon \to 0} \int_{|x| \geq \epsilon} \frac{1}{4\pi|x|} \Delta \phi(x) \, dx \), and use the divergence theorem (integrate by parts in polar coordinates).

Problem 8. On \( \mathbb{R}^3 \), write \( x = (x', x_3) \), so \( x' = (x_1, x_2) \). Suppose \( f \) is a compactly supported \( C^1 \) function in \( x_3 \geq 0 \) in \( \mathbb{R}^3 \) vanishing near \( x_3 = 0 \). Find the solution \( u \) of
\[
\Delta u = f, \quad x_3 \geq 0, \quad \partial_{x_3}u(x', 0) = 0
\]
which goes to 0 at infinity. Write your solution as explicitly in terms of \( f \) as possible.

Problem 9. (i) Using the method of reflection, solve the wave equation with Neumann boundary conditions on the interval \([0, \ell]_x\):
\[
u_{tt} - c^2u_{xx} = 0, \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x), \quad u_x(0, t) = 0 = u_x(\ell, t).
\]
You do not need to write an explicit formula containing only \( \phi \) and \( \psi \); the appropriate extension of \( \phi \) and \( \psi \) to \( \mathbb{R} \) may appear in the formula.

(ii) If \( \psi = 0 \) and \( \phi \) is \( C^\infty \) except at a point \( x_0 \in (0, \ell) \), where do you know for sure that \( u \) is \( C^\infty \)?

Problem 10. (Optional!) Show that every \( u \in \mathcal{S}'(\mathbb{R}^n) \) can be approximated by elements of \( \mathcal{S}(\mathbb{R}^n) \), i.e. show that there exist \( f_j \in \mathcal{S}(\mathbb{R}^n) \) such that \( \iota f_j \to u \) in \( \mathcal{S}'(\mathbb{R}^n) \). Here recall that \( u \to u \) in \( \mathcal{S}'(\mathbb{R}^n) \) means that \( u_j(\phi) \to u(\phi) \) for all \( \phi \in \mathcal{S}(\mathbb{R}^n) \).

Hints: it suffices to show that there is \( f_j \) and \( m \) such that \( |\iota f_j(\phi) - u(\phi)| \leq j^{-1}\|\phi\|_m \) for all \( \phi \in \mathcal{S} \), see Problem 4 for the notation. So first consider \( v_j = \chi_{j} u \), where \( \chi_j(x) = \chi(x/j) \), and \( \chi \in C_c^\infty(\mathbb{R}^n) \) is identically 1 for \( |x| < 1 \), identically 0 for \( |x| > 2 \). Thus, \( v_j \) are compactly supported distributions, and show that \( v \to u \) in \( \mathcal{S}'(\mathbb{R}^n) \) in the strong sense that for some \( \tilde{m} \), \( |v_j(\phi) - u(\phi)| \leq \tilde{C} j^{-1}\|\phi\|_{\tilde{m}} \).

Now to get the \( f_j \), we approximate the \( v_j \) as follows: \( Fv_j \) is \( C^\infty \), but does not decay at infinity. So let \( g_{jk}(\xi) = \chi_k(\xi)(Fv_j)(\xi), \chi_k \) as above. Show that \( g_{jk} \in \mathcal{S}(\mathbb{R}^n) \), and \( \iota g_{jk} \to Fv_j \) in \( \mathcal{S}'(\mathbb{R}^n) \) as \( k \to \infty \). Now consider \( F^{-1}g_{jk} \).