MATH 220: Problem Set 6
Solutions

Problem 1. g is obviously not a Schwartz function, but it is a tempered distribution, as $\iota_g(\psi)$ is finite for all $\psi \in \mathcal{S}(\mathbb{R}^3)$ (which you should check for yourself). Therefore we can make sense of the Fourier transform of the distribution ι_g.

(i) From PSET 5, Problem 4, we know that (with $n = -1$) for $a > 0$

$$\begin{aligned} (\mathcal{F}g_a)(\xi) &= 2\pi i \frac{1}{|\xi|}\left(\frac{1}{(a+i|\xi|)} - \frac{1}{(a-i|\xi|)}\right) \\ &= \frac{4\pi}{(a^2+|\xi|^2)}. \end{aligned} \tag{1}$$

First notice that $g_a(x) \to g(x)$ pointwise for all $x \neq 0$. Therefore, we have that for all $\xi \neq 0$,

$$(\mathcal{F}g)(\xi) = \left(\mathcal{F}\left(\lim_{a\to 0^+} g_a\right)\right)(\xi) \overset{\text{LDCT}}{=} \lim_{a\to 0^+} (\mathcal{F}g_a)(\xi) = \frac{4\pi}{|\xi|^2}. \tag{2}$$

Notice that both $\mathcal{F}g_a$ and $\mathcal{F}g$ define tempered distributions given by $\iota_{\mathcal{F}g_a}$ and $\iota_{\mathcal{F}g}$ respectively. This can again be done by checking (which is left to you as an exercise) that $\iota_{\mathcal{F}g_a}(\psi)$ and $\iota_{\mathcal{F}g}(\psi)$ are defined for all $\psi \in \mathcal{S}(\mathbb{R}^3)$. Next notice that for all $\psi \in \mathcal{S}(\mathbb{R}^3)$, $(\mathcal{F}\iota_g)(\psi) = \iota_g(\mathcal{F}\psi) = \iota_{\mathcal{F}g}(\psi)$, where the first equality is by definition, and the second follows by Fubini's theorem (you should check that conditions for applying Fubini's theorem hold). This proves that $\mathcal{F}\iota_g = \iota_{\mathcal{F}g}$.

(ii) From now on, we are going to use shortcuts in the notations as often as possible. Here we want to solve $\Delta u = f$ in $\mathbb{R}^3$, where $f \in \mathcal{S}(\mathbb{R}^n)$. Taking the Fourier transform on both sides, we have

$$-|\xi|^2 (\mathcal{F}u)(\xi) = (\mathcal{F}f)(\xi), \tag{3}$$

which immediately gives

$$(\mathcal{F}u)(\xi) = -\frac{(\mathcal{F}f)(\xi)}{|\xi|^2} \tag{4}$$

and taking the inverse Fourier transform:

$$u = -\mathcal{F}^{-1}\left(\frac{\mathcal{F}f}{|\xi|^2}\right) = -\mathcal{F}^{-1}\left(\frac{1}{|\xi|^2}\right) * f = -\frac{1}{4\pi|x|} * f. \tag{5}$$

That means, in integral form,

$$u(x) = -\int_{\mathbb{R}^3} \frac{f(y)}{4\pi|x-y|} dy. \tag{6}$$

Problem 2. By definition, we have that, since $u = \delta_{|x|-R}$ is compactly supported, with $g_\xi(x) = e^{-ix\cdot\xi}$,

$$\left(\mathcal{F}u\right)(\xi) = u(g_\xi) = R^2 \int_{\mathbb{S}^2} e^{-iR\omega\cdot\xi} dS(\omega). \tag{7}$$

Now using spherical coordinates centered around ξ (same procedure as in PSET 5, Problem 4), we get

$$\begin{aligned} \left(\mathcal{F}u\right)(\xi) &= R^2 \int_0^\pi \int_0^{2\pi} e^{-iR|\xi|\cos(\theta_\xi)} \sin\theta_\xi d\phi_\xi d\theta_\xi \\ &= 2\pi R^2 \int_0^\pi e^{-iR|\xi|\cos(\theta_\xi)} \sin\theta_\xi d\theta_\xi \\ &= 2\pi R \left(\frac{e^{iR|\xi|} - e^{-iR|\xi|}}{i|\xi|}\right) \\ &= 4\pi R^2 \frac{\sin(R|\xi|)}{R|\xi|}. \end{aligned} \tag{8}$$

Problem 3. We want to solve the wave equation in $\mathbb{R}^3_x \times \mathbb{R}_t$.
From the course notes, we already know that

$$u(x,t) = \mathcal{F}^{-1}\left(\cos(c|\xi|t)\right) *_x \phi + \mathcal{F}^{-1}\left(\frac{\sin(ct|\xi|)}{c|\xi|}\right) *_x \psi. \tag{9}$$

The novelty here is that we actually now know how to solve it in $3D$! Indeed, from the last problem, we know that

$$\left(\mathcal{F}\delta_{|x|-ct}\right)(\xi) \quad = 4\pi c^2 t \frac{\sin(ct|\xi|)}{c|\xi|}. \tag{10}$$

Therefore

$$\mathcal{F}^{-1}\left(\frac{\sin(ct|\xi|)}{c|\xi|}\right) \quad = \frac{1}{4\pi c^2 t}\delta_{|x|-ct}. \tag{11}$$

Moreover, since we have $\cos(ct|\xi|) = \dfrac{\partial}{\partial t}\left(\dfrac{\sin(ct|\xi|)}{c|\xi|}\right)$, we get that

$$\mathcal{F}^{-1}\left(\cos(c|\xi|t)\right) = \mathcal{F}^{-1}\left(\frac{\partial}{\partial t}\left(\frac{\sin(ct|\xi|)}{c|\xi|}\right)\right) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\delta_{|x|-ct}\right). \tag{12}$$

And finally, we get the closed form solution:

$$\begin{aligned}
u(x,t) &= \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\delta_{|x|-ct}\right) *_x \phi + \frac{1}{4\pi c^2 t}\delta_{|x|-ct} *_x \psi \\
&= \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\delta_{|x|-ct}\right)(\phi_x) + \frac{1}{4\pi c^2 t}\delta_{|x|-ct}(\psi_x) \\
&= \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int_{|y-x|=ct}\phi(y)dS(y)\right) + \frac{1}{4\pi c^2 t}\int_{|y-x|=ct}\psi(y)dS(y).
\end{aligned} \tag{13}$$

Problem 4. The last remaining step of the proof (rest done in the hint) is to verify that $\psi_j = \dfrac{\phi_j}{j\|\phi_j\|_j}$ is such that $\psi_j \xrightarrow[j\to+\infty]{} 0$ in $\mathcal{S}(\mathbb{R}^n)$. So let's do it by proving that for any α, $\beta \in \mathbb{N}^n$ multi-indices , for any $\epsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that

$$\forall j > j_0,\ \sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial^\beta(\psi_j(x))\right| < \epsilon. \tag{14}$$

So let α, $\beta \in \mathbb{N}^p$, and $\epsilon > 0$. We just need to take $j_0 \in \mathbb{N}$ such that

$$j_0 > |\alpha| + |\beta| \text{ and } \frac{1}{j_0} < \epsilon. \tag{15}$$

We then get, for all $j > j_0$,

$$\frac{|x^\alpha \partial^\beta(\phi_j(x))|}{\|\phi_j\|_j} < 1 \text{ and } \sup_{x\in\mathbb{R}^n}\left|x^\alpha \partial^\beta(\psi_j(x))\right|\frac{1}{j} < \epsilon, \tag{16}$$

where the first inequality holds by definition of $\|\cdot\|_j$. Now we get the contradiction by noticing that by continuity of u, we should get $u(\psi_j) \xrightarrow[j\to+\infty]{} 0$, but following the reasoning of the hint, $u(\psi_j) > 1$, which completes the whole proof.

Remark. We know that the converse is true. Meaning, if $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ is linear, and if there exist an integer $m \in \mathbb{N}$ and a constant $C > 0$ such that, for all $\phi \in \mathcal{S}(\mathbb{R}^n)$

$$|u(\phi)| \leq C\|\phi\|_m, \tag{17}$$

then u is continuous and therefore defines a tempered distribution. Hence, the continuity property is equivalent to (17).

Problem 5. Since $u \in \mathcal{S}'(\mathbb{R}^3)$ and has compact support, we can define the function $(\mathcal{F}u)(\xi) = u(\phi_\xi)$, with $\phi_\xi(x) = f(x)e^{-ix\cdot\xi}$, and $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ is 1 in the neighborhood of the support of u. Indeed,

$$(\mathcal{F}u)(\xi) = u(g_\xi) = u(fg_\xi) = u(\phi_\xi), \tag{18}$$

where again $g_\xi(x) = e^{-ix\cdot\xi}$. Since $u \in \mathcal{S}'(\mathbb{R}^3)$ and $\phi_\xi \in \mathcal{S}(\mathbb{R}^n)$ (because $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$) , we can use the previous problem and write that there exist some constants $m \in \mathbb{N}$ and $C_1 > 0$ such that

$$|(\mathcal{F}u)(\xi)| = |u(\phi_\xi)| \leq C_1 \|\phi_\xi\|_m. \tag{19}$$

Now since f is compactly supported, there exists a constant $C_2 > 0$ such that for all $|\alpha| \leq m, |\beta| \leq m$, $\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)| \leq C_2$. On the other hand, there exists a constant $C_3 > 0$ such that for all $|\beta| \leq m$, $|\partial^\beta e^{-ix\cdot\xi}| \leq C_3 |\xi|^\beta$. Now using, say, Leibniz formula, and considering the maximum $C > 0$ of the absolute value of the factors in front of powers of $|\xi|$, we get to the conclusion, namely

$$|(\mathcal{F}u)(\xi)| \leq C\,(1 + |\xi|)^m\,. \tag{20}$$

Problem 6. Let $u, v \in \mathcal{S}'(\mathbb{R}^3)$, and v with compact support. We can define the convolution $u * v$ in several ways. Let's see two of them.

A first definition. For any $\psi \in \mathcal{S}(\mathbb{R}^n)$, define

$$(u * v)(\psi) = u(w), \tag{21}$$

where w is a smooth function given by $w(x) = (v*\psi_-)(-x)$ and $\psi_-(z) = \psi(-z)$. Since v has compact support, $v * \psi_-$ is really in $\mathcal{S}(\mathbb{R}^n)$. Therefore the stated definition makes sense. Now let's show that it is consistent with the definition if one of them is in $\mathcal{S}(\mathbb{R}^n)$.

If $u = \iota_f$ with $f \in \mathcal{S}(\mathbb{R}^n)$ and $v \in \mathcal{S}'(\mathbb{R}^3)$ with compact support, then

$$\begin{aligned}(u * v)(\psi) &= u(w)\\ &= \iota_f(w)\\ &= \int_{\mathbb{R}^n} f(x)(v * \psi_-)(-x)dx\\ &= \int_{\mathbb{R}^n} f(x) v\left((\psi_-)_{-x}\right) dx\\ &= v\left(\int_{\mathbb{R}^n} f(x)(\psi_-)_{-x} dx\right)\\ &= v\left(\int_{\mathbb{R}^n} f(x)\psi(x+z)dx\right)\\ &= v\left(\int_{\mathbb{R}^n} f(y-z)\psi(y)dy\right)\\ &= \int_{\mathbb{R}^n} v(f_y)\psi(y)dy\\ &= \int_{\mathbb{R}^n} (v * f)(y)\psi(y)dy\\ &= (v * f)(\psi).\end{aligned} \tag{22}$$

And if $u \in \mathcal{S}'(\mathbb{R}^3)$ and $v = \iota_g$ with $f \in \mathcal{S}(\mathbb{R}^n)$ with compact support, then

$$\begin{aligned}(u * v)(\psi) &= u(w) \\ &= u\left(\int_{\mathbb{R}^n} g(x)\psi(x+z)dx\right) \\ &= u\left(\int_{\mathbb{R}^n} g(y-z)\psi(y)dy\right) \\ &= \int_{\mathbb{R}^n} u(g_y)\psi(y)dy \\ &= \int_{\mathbb{R}^n} (u * g)(y)\psi(y)dy \\ &= (u * g)(\psi).\end{aligned} \tag{23}$$

Remarks. We use w because it allows us to write a "clean" definition of the convolution without explicitly writing the dependencies on variables.
Also, consistency should be tested on "nice" objects (definitions should be made based on good objects). Therefore I think that it is enough to show consistency in the case $u = \iota_f$, $v = \iota_g$ with $f, g \in \mathcal{S}(\mathbb{R}^n)$. For this case, we have:

$$\begin{aligned}(u * v)(\psi) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(y)\psi(x+y)dxdy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(z-x)\psi(z)dxdz \\ &= \int_{\mathbb{R}^n} (f * g)(z)\psi(z)dz \\ &= (\iota_{f*g})(\psi).\end{aligned} \tag{24}$$

Notice that to get to the definition, the chain of reasoning is made in the reverse order in (24).

A second definition. We can define the convolution using Fourier transforms:

$$u * v = \mathcal{F}^{-1}\left(\mathcal{F}u \cdot \mathcal{F}v(\xi)\right), \tag{25}$$

where $\mathcal{F}u$ is the tempered distribution defining the Fourier transform of u, while $\mathcal{F}v(\xi)$ is the $\mathcal{C}^\infty$ function associated to the tempered distribution of the Fourier transform of v. We need to show why this definition makes sense. Since v has compact support, we know from the course that $\mathcal{F}v(\xi)$ is $\mathcal{C}^\infty$. We can write

$$\mathcal{F}^{-1}\left(\mathcal{F}u \cdot \mathcal{F}v(\xi)\right)(\phi) = \left(\mathcal{F}u \cdot \mathcal{F}v(\xi)\right)(\mathcal{F}^{-1}\phi) = \left(\mathcal{F}v(\xi) \cdot \mathcal{F}u\right)(\mathcal{F}^{-1}\phi) = \mathcal{F}u(\mathcal{F}v(\xi) \cdot \mathcal{F}^{-1}\phi), \tag{26}$$

where the last equation is well-defined since $\mathcal{F}v(\xi) \cdot \mathcal{F}^{-1}\phi \in \mathcal{S}(\mathbb{R}^n)$ by Problem 5. Lastly, the definition of $u * v$ is clearly consistent with the definition when one of them is in $\mathcal{S}(\mathbb{R}^n)$ (or both in $\mathcal{S}(\mathbb{R}^n)$). Both these cases were done during the office hours.

Problem 7.

(i) By definition of the fundamental solution for P, we have that $PE = \delta_0$, meaning that

$$(PE * f)(x) = (\delta_0 * f)(x) = f(x). \tag{27}$$

Therefore, to show that $Pu = f$, it is enough to show that $PE*f = P(E*f)$, and by the very form of P, actually we can restrict ourselves to $\partial_j(E*f) = \partial_j E * f$. For any $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\partial_j(E*f)(\psi) &= -(E*f)(\partial_j\psi) = -\int_{\mathbb{R}^n}(E*f)(x)\partial_j\psi(x)dx \\ &= -\int_{\mathbb{R}^n} E(f_x)\partial_j\psi(x)dx = -E\left(\int_{\mathbb{R}^n} f_x\partial_j\psi(x)dx\right) \\ &= E\left(\int_{\mathbb{R}^n} -\partial_j f(x-y)\psi(x)dx\right) = \int_{\mathbb{R}^n} E\left(-\partial_j f_x\right)\psi(x)dx \\ &= \int_{\mathbb{R}^n}\partial_j E\left(f_x\right)\psi(x)dx = \int_{\mathbb{R}^n}(\partial_j E * f)(x)\psi(x)dx \\ &= (\partial_j E * f)(\psi),\end{aligned} \tag{28}$$

which ends the proof.

(ii) We want to show here that $P(E*f)(\psi) = f(\psi)$.
Using the definition of the convolution used in Problem 6 with Fourier transform , we get that

$$\begin{aligned}\partial_j(E*f)(\psi) &= -(E*f)(\partial_j\psi) = -\mathcal{F}^{-1}\left(\mathcal{F}E\cdot\mathcal{F}f(\xi)\right)(\partial_j\phi) \\ &= -\left(\mathcal{F}E\cdot\mathcal{F}f(\xi)\right)(\mathcal{F}^{-1}\partial_j\phi) = -\left(\mathcal{F}E\cdot\mathcal{F}f(\xi)\right)(-i\xi_j\mathcal{F}^{-1}\phi) \\ &= i\xi_j\left(\mathcal{F}E\cdot\mathcal{F}f(\xi)\right)(\mathcal{F}^{-1}\phi) = \mathcal{F}^{-1}\left(i\xi_j\mathcal{F}E\cdot\mathcal{F}f(\xi)\right)(\phi) \\ &= \mathcal{F}^{-1}\left(\mathcal{F}\partial_j E\cdot\mathcal{F}f(\xi)\right)(\phi) \\ &= (\partial_j E * f)(\psi).\end{aligned} \tag{29}$$

We can work this out similarly using the other definition.

(iii) By using the Fourier transform, the result is straightforward. Indeed, we want to find E such that $\Delta E = \delta_0$. So we take the Fourier transform on both sides to get

$$-|\xi|^2\mathcal{F}E = \mathcal{F}\delta_0 = 1. \tag{30}$$

Therefore $\mathcal{F}E = -\dfrac{1}{4\pi|\xi|^2}$ and by using Problem 1, we know that E is given by

$$E(x) = -\frac{1}{4\pi|x|}. \tag{31}$$

Now by using direct calculations, we are going to ue the hint. So here we take the function $E(x) = -\frac{1}{4\pi|x|}$, and verify that it is a fundamental solution for Δ in $\mathbb{R}^3$, that is $\Delta E(\phi) = \delta_0(\phi) = \phi(0)$. We know that for all $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\Delta E(\phi) = E(\Delta\phi)$. Since we have a singularity at 0 for E, we need to be careful and work with limits.

Take $\epsilon > 0$. By the divergence theorem, we obtain that

$$I_\epsilon = \int_{|x|\geq\epsilon} -\frac{1}{4\pi|x|}\Delta\phi(x)dx = -\int_{|x|\geq\epsilon} \nabla\left(-\frac{1}{4\pi|x|}\right)\cdot\nabla\phi(x)dx + \int_{|x|=\epsilon} -\frac{1}{4\pi|x|}\frac{\partial\phi}{\partial\nu}dS(x), \tag{32}$$

where ν is the outward normal (pointing towards origin) along $\partial B(0,\epsilon)$, ball centered at 0 and of radius ϵ. Applying divergence theorem (again) to the first term in the right hand side of the above equation, we get

$$I_\epsilon = \int_{|x|\geq\epsilon} \Delta\left(-\frac{1}{4\pi|x|}\right)\phi(x)dx - \int_{|x|=\epsilon} \frac{\partial\left(-\frac{1}{4\pi|x|}\right)}{\partial\nu}\phi(x)dS(x) + \int_{|x|=\epsilon} -\frac{1}{4\pi|x|}\frac{\partial\phi}{\partial\nu}dS(x). \tag{33}$$

Since $\Delta\left(-\frac{1}{4\pi|x|}\right) = 0$ for $x \neq 0$, the first term on the right hand side just vanishes. Since $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, using spherical coordinates, we obtain

$$\begin{aligned} \left|\int_{|x|=\epsilon} -\frac{1}{4\pi|x|}\frac{\partial\phi}{\partial\nu}dS(x)\right| &\leq C\int_{|x|=\epsilon}\frac{1}{|x|}dS(x) \\ &\leq \frac{C}{\epsilon}\int_{|x|=\epsilon} dS(x) \\ &= \tfrac{C}{\epsilon}4\pi\epsilon^2 = 4\pi C\epsilon \xrightarrow[\epsilon\to 0]{} 0. \end{aligned} \tag{34}$$

Therefore we also get rid to this term at the limit. Finally, we need to show that

$$\lim_{\epsilon\to 0^+}\int_{|x|=\epsilon} \frac{\partial\left(\frac{1}{4\pi|x|}\right)}{\partial\nu}\phi(x)dS(x) = \phi(0). \tag{35}$$

This is done by first noticing that

$$\frac{\partial\left(-\frac{1}{4\pi|x|}\right)}{\partial\nu} = \nabla\left(-\frac{1}{4\pi|x|}\right)\cdot\nu = \frac{1}{4\pi}\frac{x}{|x|^3}\cdot\frac{-x}{|x|} = -\frac{1}{4\pi|x|^2}. \tag{36}$$

Therefore

$$\lim_{\epsilon\to 0^+}\int_{|x|=\epsilon} \frac{\partial\left(-\frac{1}{4\pi|x|}\right)}{\partial\nu}\phi(x)dS(x) = \lim_{\epsilon\to 0^+}\int_{|x|=\epsilon} -\frac{1}{4\pi\epsilon^2}\phi(x)dS(x) = -\phi(0) \tag{37}$$

The limit here is easy to justify since for instance we can write

$$\begin{gathered}\inf_{y\in\mathbb{S}^2}\phi(\epsilon y)\int_{|x|=\epsilon}\frac{1}{4\pi\epsilon^2}dS(x)\leq\int_{|x|=\epsilon}\frac{1}{4\pi\epsilon^2}\phi(x)dS(x)\leq\sup_{y\in\mathbb{S}^2}\phi(\epsilon y)\int_{|x|=\epsilon}\frac{1}{4\pi\epsilon^2}dS(x)\\ \Updownarrow\\ \inf_{y\in\mathbb{S}^2}\phi(\epsilon y)\leq\int_{|x|=\epsilon}\frac{1}{4\pi\epsilon^2}\phi(x)dS(x)\leq\sup_{y\in\mathbb{S}^2}\phi(\epsilon y)\end{gathered}\tag{38}$$

and

The limit here is easy to justify since for instance we can write

$$\begin{cases}\inf_{y\in\mathbb{S}^2}\phi(\epsilon y)\xrightarrow[\epsilon\to 0]{}\phi(0),\\ \sup_{y\in\mathbb{S}^2}\phi(\epsilon y)\xrightarrow[\epsilon\to 0]{}\phi(0).\end{cases}\tag{39}$$

Hence

$$E(\Delta\phi)=\lim_{\epsilon\to 0^+}I_\epsilon=\lim_{\epsilon\to 0^+}\int_{|x|\geq\epsilon}\Delta\left(-\frac{1}{4\pi|x|}\right)\phi(x)dx=\phi(0).\tag{40}$$

Problem 8. We follow the hint. Let's consider $v_j=\chi_j u$ (they are hence of compact support). We want to show that

$$v_j\xrightarrow[j\to+\infty]{}u\text{ in }\mathcal{S}'(\mathbb{R}^3).\tag{41}$$

But this means that, for all $\psi\in\mathcal{S}(\mathbb{R}^n)$,

$$v_j(\psi)\xrightarrow[j\to+\infty]{}u(\psi),\tag{42}$$

or, defining $\psi_j=\chi_j\psi$,

$$v_j(\psi)=\chi_j u(\psi)=u(\psi_j)\xrightarrow[j\to+\infty]{}u(\psi),\tag{43}$$

Therefore, it suffices to show that $\psi_j\xrightarrow[j\to+\infty]{}\psi$ in $\mathcal{S}(\mathbb{R}^n)$. Indeed, for all $j\leq 1$, ψ_j is in $\mathcal{S}(\mathbb{R}^n)$. Therefore $\psi-\chi_j\psi=(1-\chi_j)\psi$ is in $\mathcal{S}(\mathbb{R}^n)$ and also $Supp((1-\chi_j)\psi)\subset\{|x|\geq j\}$. Now let's take $\alpha,\beta\in\mathbb{N}^p$ (p arbitrary) multi-indices. Since by definition of χ_j, $|\partial^\beta\chi_j(x)|\leq|\partial^\beta\chi(x)|\leq C_1$, then using the fact that ψ is a Schwartz function (along with the Leibniz formula for instance) we conclude that

$$|x^\alpha\partial^\beta(1-\chi_j(x))\psi(x)|\leq\sup_{\{|x|\geq j\}}\frac{C}{1+|x|}\leq\frac{C}{1+j}\xrightarrow[j\to+\infty]{}0,\tag{44}$$

which ends the proof of the first fact.

Now since v_j are compactly supported, we know that $\mathcal{F}v_j$ is $\mathcal{C}^\infty$, but we are not assured that they rapidly decay at infinity. Therefore we define

$$g_{jk}(\xi)=\chi_k(\xi)\left(\mathcal{F}v_j\right)(\xi)\tag{45}$$

Since v_j has compact support and $\chi_j \in \mathcal{S}(\mathbb{R}^n)$, Pb5 allows us to conclude that $g_{jk} \in \mathcal{S}(\mathbb{R}^n)$ (or in this case, we can simply invoke the fact that $\chi_j \in \mathcal{C}_c^\infty(\mathbb{R}^n)$). Now using exactly the same proof as for the first step, we get that

$$\iota_{g_{jk}} \xrightarrow[j\to+\infty]{} \mathcal{F}v_j \text{ in } \mathcal{S}'(\mathbb{R}^3). \tag{46}$$

Finally, considering $h_{jk} = \mathcal{F}^{-1}g_{jk}$, a first step limit gives

$$h_{jk} = \mathcal{F}^{-1}g_{jk} \xrightarrow[j\to+\infty]{} \mathcal{F}^{-1}\left(\mathcal{F}v_j\right) = v_j \text{ in } \mathcal{S}'(\mathbb{R}^3), \tag{47}$$

and a second step (first result) ends the proof:

$$v_j \xrightarrow[j\to+\infty]{} u \text{ in } \mathcal{S}'(\mathbb{R}^3). \tag{48}$$