Problem 1. \( g \) is obviously not a Schwartz function, but it is a tempered distribution, as \( \iota_g(\psi) \) is finite for all \( \psi \in \mathcal{S}(\mathbb{R}^3) \) (which you should check for yourself). Therefore we can make sense of the Fourier transform of the distribution \( \iota_g \).

(i) From PSET 5, Problem 4, we know that (with \( n = -1 \)) for \( a > 0 \)

\[
(Fg_a)(\xi) = 2\pi i \frac{1}{4\pi} \left( \frac{1}{(a+i|\xi|)} - \frac{1}{(a-i|\xi|)} \right)
\]

\[= \frac{4\pi}{(a^2 + |\xi|^2)}. \tag{1}\]

First notice that \( g_a(x) \rightarrow g(x) \) pointwise for all \( x \neq 0 \). Therefore, we have that for all \( \xi \neq 0 \),

\[(Fg)(\xi) = \left( F \left( \lim_{a \to 0^+} g_a \right) \right)(\xi) \overset{\text{LDCT}}{=} \lim_{a \to 0^+} (Fg_a)(\xi) = \frac{4\pi}{|\xi|^2}. \tag{2}\]

Notice that both \( Fg_a \) and \( Fg \) define tempered distributions given by \( \iota_{Fg_a} \) and \( \iota_{Fg} \) respectively. This can again be done by checking (which is left to you as an exercise) that \( \iota_{Fg_a}(\psi) \) and \( \iota_{Fg}(\psi) \) are defined for all \( \psi \in \mathcal{S}(\mathbb{R}^3) \). Next notice that for all \( \psi \in \mathcal{S}(\mathbb{R}^3) \), \( (F\iota_g)(\psi) = \iota_g(F\psi) = \iota_{Fg}(\psi) \), where the first equality is by definition, and the second follows by Fubini’s theorem (you should check that conditions for applying Fubini’s theorem hold). This proves that \( F\iota_g = \iota_{Fg} \).

(ii) From now on, we are going to use shortcuts in the notations as often as possible. Here we want to solve \( \Delta u = f \) in \( \mathbb{R}^3 \), where \( f \in \mathcal{S}(\mathbb{R}^n) \). Taking the Fourier transform on both sides, we have

\[-|\xi|^2 (Fu)(\xi) = (Ff)(\xi), \tag{3}\]

which immediately gives

\[(Fu)(\xi) = -\frac{(Ff)(\xi)}{|\xi|^2} \tag{4}\]

and taking the inverse Fourier transform:

\[u = -F^{-1} \left( \frac{Ff}{|\xi|^2} \right) = -F^{-1} \left( \frac{1}{|\xi|^2} \right) * f = -\frac{1}{4\pi|x|} * f. \tag{5}\]
That means, in integral form,

\[ u(x) = -\int_{\mathbb{R}^3} \frac{f(y)}{4\pi |x - y|} \, dy. \]  

(6)

**Problem 2.** By definition, we have that, since \( u = \delta_{|x|-R} \) is compactly supported, with \( g_\xi(x) = e^{-ix\cdot\xi} \),

\[ (\mathcal{F}u)(\xi) = u(g_\xi) = R^2 \int_{\mathbb{S}^2} e^{-iR\omega\cdot\xi} dS(\omega). \]  

(7)

Now using spherical coordinates centered around \( \xi \) (same procedure as in PSET 5, Problem 4), we get

\[ (\mathcal{F}u)(\xi) = R^2 \int_0^\pi \int_0^{2\pi} e^{-iR|\xi|\cos(\theta_\xi)} \sin \theta_\xi d\phi_\xi d\theta_\xi \]
\[ = 2\pi R^2 \int_0^{2\pi} e^{-iR|\xi|\cos(\theta_\xi)} \sin \theta_\xi d\theta_\xi \]
\[ = 2\pi R \left( \frac{e^{iR|\xi|} - e^{-iR|\xi|}}{i|\xi|} \right) \]
\[ = 4\pi R^2 \sin(R|\xi|). \]  

(8)

**Problem 3.** We want to solve the wave equation in \( \mathbb{R}^3_+ \times \mathbb{R}_+ \).

From the course notes, we already know that

\[ u(x, t) = \mathcal{F}^{-1} \left( \cos(c|\xi|t) \right) *_x \phi + \mathcal{F}^{-1} \left( \frac{\sin(c|\xi|)}{c|\xi|} \right) *_x \psi. \]  

(9)

The novelty here is that we actually now know how to solve it in 3D! Indeed, from the last problem, we know that

\[ (\mathcal{F}\delta_{|x|-ct})(\xi) = 4\pi c^2 t \frac{\sin(\xi|\xi|)}{c|\xi|}. \]  

(10)

Therefore

\[ \mathcal{F}^{-1} \left( \frac{\sin(ct|\xi|)}{c|\xi|} \right) = \frac{1}{4\pi c^2 t} \delta_{|x|-ct}. \]  

(11)

Moreover, since we have \( \cos(ct|\xi|) = \frac{\partial}{\partial t} \left( \frac{\sin(ct|\xi|)}{c|\xi|} \right) \), we get that

\[ \mathcal{F}^{-1} (\cos(c|\xi|t)) = \mathcal{F}^{-1} \left( \frac{\partial}{\partial t} \left( \frac{\sin(ct|\xi|)}{c|\xi|} \right) \right) = \frac{\partial}{\partial t} \left( \frac{1}{4\pi c^2 t} \delta_{|x|-ct} \right). \]  

(12)
And finally, we get the closed form solution:

\[
\begin{align*}
    u(x,t) &= \frac{\partial}{\partial t} \left( \frac{1}{4\pi c^2 t} \delta_{|x|-ct} \right) *_{x} \phi + \frac{1}{4\pi c^2 t} \delta_{|x|-ct} *_{x} \psi \\
    &= \frac{\partial}{\partial t} \left( \frac{1}{4\pi c^2 t} \delta_{|x|-ct} \right) (\phi_x) + \frac{1}{4\pi c^2 t} \delta_{|x|-ct} (\psi_x) \\
    &= \frac{\partial}{\partial t} \left( \frac{1}{4\pi c^2 t} \int_{|y-x|=ct} \phi(y) dS(y) \right) + \frac{1}{4\pi c^2 t} \int_{|y-x|=ct} \psi(y) dS(y).
\end{align*}
\]  

(13)

**Problem 4.** The last remaining step of the proof (rest done in the hint) is to verify that \( \psi_j = \frac{\phi_j}{\|\phi_j\|_j} \) is such that \( \psi_j \xrightarrow{j \to +\infty} 0 \) in \( S(\mathbb{R}^n) \). So let’s do it by proving that for any \( \alpha, \beta \in \mathbb{N}^n \) multi-indices , for any \( \epsilon > 0 \), there exists \( j_0 \in \mathbb{N} \) such that

\[
\forall j > j_0, \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta (\psi_j(x))| < \epsilon.
\]  

(14)

So let \( \alpha, \beta \in \mathbb{N}^p, \) and \( \epsilon > 0 \). We just need to take \( j_0 \in \mathbb{N} \) such that

\[
j_0 > |\alpha| + |\beta| \text{ and } \frac{1}{j_0} < \epsilon.
\]  

(15)

We then get, for all \( j > j_0 \),

\[
\frac{|x^\alpha \partial^\beta (\phi_j(x))|}{\|\phi_j\|_j} < 1 \text{ and } \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta (\psi_j(x))| \frac{1}{j} < \epsilon,
\]  

(16)

where the first inequality holds by definition of \( \| \cdot \|_j \). Now we get the contradiction by noticing that by continuity of \( u \), we should get \( u(\psi_j) \xrightarrow{j \to +\infty} 0 \), but following the reasoning of the hint, \( u(\psi_j) > 1 \), which completes the whole proof.

**Remark.** We know that the converse is true. Meaning, if \( u : S(\mathbb{R}^n) \to C \) is linear, and if there exist an integer \( m \in \mathbb{N} \) and a constant \( C > 0 \) such that, for all \( \phi \in S(\mathbb{R}^n) \)

\[
|u(\phi)| \leq C \|\phi\|_m,
\]  

(17)

then \( u \) is continuous and therefore defines a tempered distribution. Hence, the continuity property is equivalent to (17).

**Problem 5.** Since \( u \in S'(\mathbb{R}^3) \) and has compact support, we can define the function \( (\mathcal{F}u)(\xi) = u(\phi_\xi) \), with \( \phi_\xi(x) = f(x)e^{-ix\cdot\xi} \), and \( f \in C_\infty(\mathbb{R}^n) \) is 1 in the neighborhood of the support of \( u \). Indeed,

\[
(\mathcal{F}u)(\xi) = u(g_\xi) = u(fg_\xi) = u(\phi_\xi),
\]  

(18)
where again \( g_\xi(x) = e^{-ix\cdot\xi} \). Since \( u \in \mathcal{S}'(\mathbb{R}^3) \) and \( \phi_\xi \in \mathcal{S}(\mathbb{R}^3) \) (because \( f \in \mathcal{C}_c^\infty(\mathbb{R}^n) \)), we can use the previous problem and write that there exist some constants \( m \in \mathbb{N} \) and \( C_1 > 0 \) such that

\[
\| (\mathcal{F}u)(\xi) \| = \| u(\phi_\xi) \| \leq C_1 \| \phi_\xi \|_m.
\] (19)

Now since \( f \) is compactly supported, there exists a constant \( C_2 > 0 \) such that for all \( |\alpha| \leq m, |\beta| \leq m \),

\[
\sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)| \leq C_2.
\]

On the other hand, there exists a constant \( C_3 > 0 \) such that for all \( |\beta| \leq m \),

\[
|\partial^\beta e^{-ix\cdot\xi}| \leq C_3 |\xi|^\beta.
\]

Now using, say, Leibniz formula, and considering the maximum \( C > 0 \) of the absolute value of the factors in front of powers of \( |\xi| \), we get to the conclusion, namely

\[
\| (\mathcal{F}u)(\xi) \| \leq C (1 + |\xi|)^m.
\] (20)

**Problem 6.** Let \( u, v \in \mathcal{S}'(\mathbb{R}^3) \), and \( v \) with compact support. We can define the convolution \( u \ast v \) in several ways. Let’s see two of them.

**A first definition.** For any \( \psi \in \mathcal{S}(\mathbb{R}^n) \), define

\[
(u \ast v)(\psi) = u(w),
\] (21)

where \( w \) is a smooth function given by \( w(x) = (v \ast \psi_\cdot)(-x) \) and \( \psi_\cdot(z) = \psi(-z) \). Since \( v \) has compact support, \( v \ast \psi_\cdot \) is really in \( \mathcal{S}(\mathbb{R}^n) \). Therefore the stated definition makes sense. Now let’s show that it is consistent with the definition if one of them is in \( \mathcal{S}(\mathbb{R}^n) \).

If \( u = \iota_f \) with \( f \in \mathcal{S}(\mathbb{R}^n) \) and \( v \in \mathcal{S}'(\mathbb{R}^3) \) with compact support, then

\[
(u \ast v)(\psi) = u(w) = \iota_f(w) = \int_{\mathbb{R}^n} f(x)(v \ast \psi_\cdot)(-x)dx = \int_{\mathbb{R}^n} f(x)v((\psi_\cdot)_\cdot x)dx = \psi \left( \int_{\mathbb{R}^n} f(x)(\psi_\cdot)_\cdot x dx \right)
\]

\[
= \psi \left( \int_{\mathbb{R}^n} f(x)\psi(x + z)dx \right)
\]

\[
= \psi \left( \int_{\mathbb{R}^n} f(y - z)\psi(y)dy \right)
\]

\[
= \int_{\mathbb{R}^n} v(f_y)\psi(y)dy = \int_{\mathbb{R}^n} (v \ast f)(y)\psi(y)dy = (v \ast f)(\psi).
\] (22)
And if $u \in \mathcal{S}'(\mathbb{R}^3)$ and $v = \iota_g$ with $f \in \mathcal{S}(\mathbb{R}^n)$ with compact support, then

\[
(u * v)(\psi) = u(w) = u \left( \int_{\mathbb{R}^3} g(x) \psi(x+z)dx \right) = u \left( \int_{\mathbb{R}^n} g(y-z) \psi(y)dy \right) = \int_{\mathbb{R}^n} u(g_y) \psi(y)dy = \int_{\mathbb{R}^n} (u * g)(y) \psi(y)dy = (u * g)(\psi). \tag{23}
\]

**Remarks.** We use $w$ because it allows us to write a "clean" definition of the convolution without explicitly writing the dependencies on variables. Also, consistency should be tested on "nice" objects (definitions should be made based on good objects). Therefore I think that it is enough to show consistency in the case $u = \iota_f, v = \iota_g$ with $f, g \in \mathcal{S}(\mathbb{R}^n)$. For this case, we have:

\[
(u * v)(\psi) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)g(y) \psi(x+y)dxdy = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)g(z-x) \psi(z)dxdz = \int_{\mathbb{R}^n} (f * g)(z) \psi(z)dz = (\iota_{f*g})(\psi). \tag{24}
\]

Notice that to get to the definition, the chain of reasoning is made in the reverse order in (24).

**A second definition.** We can define the convolution using Fourier transforms:

\[
u * v = \mathcal{F}^{-1} (\mathcal{F}u \cdot \mathcal{F}v(\xi)), \tag{25}
\]

where $\mathcal{F}u$ is the tempered distribution defining the Fourier transform of $u$, while $\mathcal{F}v(\xi)$ is the $C^\infty$ function associated to the tempered distribution of the Fourier transform of $v$. We need to show why this definition makes sense. Since $v$ has compact support, we know from the course that $\mathcal{F}v(\xi)$ is $C^\infty$. We can write

\[
\mathcal{F}^{-1} (\mathcal{F}u \cdot \mathcal{F}v(\xi)) \phi = (\mathcal{F}u \cdot \mathcal{F}v(\xi)) (\mathcal{F}^{-1} \phi) = (\mathcal{F}v(\xi) \cdot \mathcal{F}u) (\mathcal{F}^{-1} \phi) = \mathcal{F}u(\mathcal{F}v(\xi) \cdot \mathcal{F}^{-1} \phi), \tag{26}
\]

where the last equation is well-defined since $\mathcal{F}v(\xi) \cdot \mathcal{F}^{-1} \phi \in \mathcal{S}(\mathbb{R}^n)$ by Problem 5. Lastly, the definition of $u * v$ is clearly consistent with the definition when one of them is in $\mathcal{S}(\mathbb{R}^n)$ (or both in $\mathcal{S}(\mathbb{R}^n)$). Both these cases were done during the office hours.
Problem 7.

(i) By definition of the fundamental solution for $P$, we have that
$$ (PE * f)(x) = (\delta_0 * f)(x) = f(x). \quad (27) $$
Therefore, to show that $Pu = f$, it is enough to show that $PE * f = P(E * f)$, and by the very form of $P$, actually we can restrict ourselves to
$$ \partial_j (E * f) = \partial_j E * f. $$
For any $\psi \in C^\infty_c(\mathbb{R}^n)$,
$$ \partial_j (E * f)(\psi) = -\partial_j f(x) \psi(x)dx = - \int_{\mathbb{R}^n} (E * f)(x) \partial_j \psi(x)dx = - \int_{\mathbb{R}^n} E(f_x) \partial_j \psi(x)dx = - E \left( \int_{\mathbb{R}^n} f_x \partial_j \psi(x)dx \right) = \int_{\mathbb{R}^n} \partial_j (E * f)(x) \psi(x)dx = (\partial_j E * f)(\psi), \quad (28) $$
which ends the proof.

(ii) We want to show here that $P(E * f)(\psi) = f(\psi)$.
Using the definition of the convolution used in Problem 6 with Fourier transform, we get that
$$ \partial_j (E * f)(\psi) = -(E * f)(\partial_j \psi) = - \int_{\mathbb{R}^n} (E * f)(x) \partial_j \psi(x)dx = - \int_{\mathbb{R}^n} E(f_x) \partial_j \psi(x)dx = - E \left( \int_{\mathbb{R}^n} f_x \partial_j \psi(x)dx \right) = \int_{\mathbb{R}^n} \partial_j (E * f)(x) \psi(x)dx = (\partial_j E * f)(\psi). \quad (29) $$
We can work this out similarly using the other definition.

(iii) By using the Fourier transform, the result is straightforward. Indeed, we want to find $E$ such that $\Delta E = \delta_0$. So we take the Fourier transform on both sides to get
$$ -|\xi|^2 \mathcal{F}E = \mathcal{F}\delta_0 = 1. \quad (30) $$
Therefore $\mathcal{F}E = - \frac{1}{4\pi |\xi|^2}$ and by using Problem 1, we know that $E$ is given by
$$ E(x) = - \frac{1}{4\pi |x|}. \quad (31) $$
Now by using direct calculations, we are going to use the hint. So here we take the function \( E(x) = -\frac{1}{4\pi|x|} \), and verify that it is a fundamental solution for \( \Delta \) in \( \mathbb{R}^3 \), that is \( \Delta E(\phi) = \delta_0(\phi) = \phi(0) \). We know that for all \( \phi \in C_0^\infty(\mathbb{R}^n) \), \( \Delta E(\phi) = E(\Delta \phi) \). Since we have a singularity at 0 for \( E \), we need to be careful and work with limits.

Take \( \epsilon > 0 \). By the divergence theorem, we obtain that

\[
I_\epsilon = \int_{|x| \geq \epsilon} -\frac{1}{4\pi|x|} \Delta \phi(x) dx = -\int_{|x| \geq \epsilon} \nabla \left( -\frac{1}{4\pi|x|} \right) \cdot \nabla \phi(x) dx + \int_{|x| = \epsilon} -\frac{1}{4\pi|x|} \frac{\partial \phi}{\partial \nu} dS(x),
\]

where \( \nu \) is the outward normal (pointing towards origin) along \( \partial B(0, \epsilon) \), ball centered at 0 and of radius \( \epsilon \). Applying divergence theorem (again) to the first term in the right hand side of the above equation, we get

\[
I_\epsilon = \int_{|x| \geq \epsilon} \Delta \left( -\frac{1}{4\pi|x|} \right) \phi(x) dx - \int_{|x| = \epsilon} \frac{\partial }{\partial \nu} \left( -\frac{1}{4\pi|x|} \right) \phi(x) dS(x) + \int_{|x| = \epsilon} -\frac{1}{4\pi|x|} \frac{\partial \phi}{\partial \nu} dS(x).
\]

Since \( \Delta \left( -\frac{1}{4\pi|x|} \right) = 0 \) for \( x \neq 0 \), the first term on the right hand side just vanishes. Since \( \phi \in C_0^\infty(\mathbb{R}^n) \), using spherical coordinates, we obtain

\[
\left| \int_{|x| = \epsilon} -\frac{1}{4\pi|x|} \frac{\partial \phi}{\partial \nu} dS(x) \right| \leq C \int_{|x| = \epsilon} \frac{1}{|x|} dS(x)
\leq C \int_{|x| = \epsilon} dS(x)
= \frac{C}{\epsilon} 4\pi \epsilon^2 = 4\pi C \epsilon \to 0 \quad (34)
\]

Therefore we also get rid to this term at the limit. Finally, we need to show that

\[
\lim_{\epsilon \to 0^+} \int_{|x| = \epsilon} \frac{\partial }{\partial \nu} \left( \frac{1}{4\pi|x|} \right) \phi(x) dS(x) = \phi(0). \quad (35)
\]

This is done by first noticing that

\[
\frac{\partial }{\partial \nu} \left( \frac{1}{4\pi|x|} \right) = \nabla \left( \frac{1}{4\pi|x|} \right) \cdot \nu = \frac{1}{4\pi \frac{x}{|x|^2}} \frac{-x}{|x|} = -\frac{1}{4\pi|x|^2}. \quad (36)
\]

Therefore

\[
\lim_{\epsilon \to 0^+} \int_{|x| = \epsilon} \frac{\partial }{\partial \nu} \left( \frac{1}{4\pi|x|} \right) \phi(x) dS(x) = \lim_{\epsilon \to 0^+} \int_{|x| = \epsilon} -\frac{1}{4\pi\epsilon^2} \phi(x) dS(x) = -\phi(0). \quad (37)
\]
The limit here is easy to justify since for instance we can write
\[
\inf_{y \in S^2} \phi(\epsilon y) \int_{|x| = \epsilon} \frac{1}{4\pi \epsilon^2} dS(x) \leq \int_{|x| = \epsilon} \frac{1}{4\pi \epsilon^2} \phi(x) dS(x) \leq \sup_{y \in S^2} \phi(\epsilon y) \int_{|x| = \epsilon} \frac{1}{4\pi \epsilon^2} dS(x)
\]
and
\[
\inf_{y \in S^2} \phi(\epsilon y) \leq \int_{|x| = \epsilon} \frac{1}{4\pi \epsilon^2} \phi(x) dS(x) \leq \sup_{y \in S^2} \phi(\epsilon y)
\]
(38)

The limit here is easy to justify since for instance we can write
\[
\begin{cases}
\inf_{y \in S^2} \phi(\epsilon y) \xrightarrow{\epsilon \to 0^+} \phi(0), \\
\sup_{y \in S^2} \phi(\epsilon y) \xrightarrow{\epsilon \to 0^+} \phi(0).
\end{cases}
\]
(39)

Hence
\[
E(\Delta \phi) = \lim_{\epsilon \to 0^+} I_{\epsilon} = \lim_{\epsilon \to 0^+} \int_{|x| \geq \epsilon} \Delta \left( -\frac{1}{4\pi |x|} \right) \phi(x) dx = \phi(0).
\]
(40)

**Problem 8.** We follow the hint. Let’s consider \( v_j = \chi_j u \) (they are hence of compact support). We want to show that
\[
v_j \xrightarrow{j \to +\infty} u \quad \text{in} \quad S'(\mathbb{R}^3).
\]
(41)

But this means that, for all \( \psi \in S(\mathbb{R}^n) \),
\[
v_j(\psi) \xrightarrow{j \to +\infty} u(\psi),
\]
(42)
or, defining \( \psi_j = \chi_j \psi \),
\[
v_j(\psi) = \chi_j u(\psi) = u(\psi_j) \xrightarrow{j \to +\infty} u(\psi),
\]
(43)

Therefore, it suffices to show that \( \psi_j \xrightarrow{j \to +\infty} \psi \) in \( S(\mathbb{R}^n) \). Indeed, for all \( j \leq 1, \psi_j \) is in \( S(\mathbb{R}^n) \). Therefore \( \psi - \chi_j \psi = (1 - \chi_j) \psi \) is in \( S(\mathbb{R}^n) \) and also \( \text{Supp}(1 - \chi_j) \psi \subset \{ |x| \geq j \} \). Now let’s take \( \alpha, \beta \in \mathbb{N}^p \) (\( p \) arbitrary) multi-indices. Since by definition of \( \chi_j \), \( |\partial^\beta \chi_j(x)| \leq |\partial^\beta \chi(x)| \leq C_1 \), then using the fact that \( \psi \) is a Schwartz function (along with the Leibniz formula for instance) we conclude that
\[
|x^\alpha \partial^\beta (1 - \chi_j)(x)| \psi(x)| \leq \sup_{|x| \geq j} \frac{C}{1 + |x|} \leq \frac{C}{1 + j} \to 0 \quad \text{as} \quad j \to +\infty,
\]
(44)

which ends the proof of the first fact.

Now since \( v_j \) are compactly supported, we know that \( \mathcal{F} v_j \) is \( C^\infty \), but we are not assured that they rapidly decay at infinity. Therefore we define
\[
g_{jk}(\xi) = \chi_k(\xi)(\mathcal{F} v_j)(\xi)
\]
(45)
Since $v_j$ has compact support and $\chi_j \in \mathcal{S}(\mathbb{R}^n)$, (4) allows us to conclude that $g_{jk} \in \mathcal{S}(\mathbb{R}^n)$ (or in this case, we can simply invoke the fact that $\chi_j \in \mathcal{C}_c^\infty(\mathbb{R}^n)$).

Now using exactly the same proof as for the first step, we get that

$$
\iota_{g_{jk}} \xrightarrow{j \to +\infty} \mathcal{F}v_j \text{ in } \mathcal{S}'(\mathbb{R}^3).
$$

Finally, considering $h_{jk} = \mathcal{F}^{-1}g_{jk}$, a first step limit gives

$$
\begin{align*}
\mathcal{F}^{-1}g_{jk} \xrightarrow{j \to +\infty} \mathcal{F}^{-1}(\mathcal{F}v_j) = v_j \text{ in } \mathcal{S}'(\mathbb{R}^3),
\end{align*}
$$

and a second step (first result) ends the proof:

$$
\begin{align*}
v_j \xrightarrow{j \to +\infty} u \text{ in } \mathcal{S}'(\mathbb{R}^3).
\end{align*}
$$