MATH 220: Problem Set 7
Solutions

Problem 1. Since $\partial_{x_3} u(x', 0) = 0$, we consider the even extension of f. Let

$$\tilde{f}(x', x_3) = \begin{cases} f(x', x_3) & \text{if } x_3 \geq 0, \\ f(x', -x_3) & \text{if } x_3 < 0, \end{cases} \tag{1}$$

Let v be a solution of $\Delta v = \tilde{f}$ in $\mathbb{R}^3$. Then from PSET 6, we know the solution, namely:

$$v(x) = \int_{\mathbb{R}^3} -\frac{1}{4\pi |x - y|} \tilde{f}(y) dy. \tag{2}$$

Now we just take $u(x', x_3) = v(x', x_3)$ when $x_3 \geq 0$.
Moreover, since $\tilde{f}$ is a compactly supported function, for $|x|$ big enough, $x \notin Supp(\tilde{f})$. Therefore, for $|x|$ big enough,

$$|v(x)| \leq \frac{1}{4\pi} \sup_{z \in Supp(\tilde{f})} \frac{1}{|x - z|} \int_{Supp(\tilde{f})} |\tilde{f}(y)| dy \leq C \sup_{z \in Supp(\tilde{f})} \frac{1}{|x - z|} \xrightarrow[|x| \to +\infty]{} 0, \tag{3}$$

and we indeed have $v(x) \xrightarrow[|x| \to +\infty]{} 0$.

Problem 2.

(i) By the method of reflection, we construct an extension of ϕ (resp. ψ) to the entire real line as follows:

$$\tilde{\phi}(x) = \begin{cases} \phi(x - 2kl) & \text{if } 2kl \leq x < (2k+1)l \text{ for some } k \in \mathbb{Z} \\ \phi(2kl - x) & \text{if } (2k-1)l \leq x < 2kl \text{ for some } k \in \mathbb{Z} \end{cases} \tag{4}$$

and similarly

$$\tilde{\psi}(x) = \begin{cases} \psi(x - 2kl) & \text{if } 2kl \leq x < (2k+1)l \text{ for some } k \in \mathbb{Z} \\ \psi(2kl - x) & \text{if } (2k-1)l \leq x < 2kl \text{ for some } k \in \mathbb{Z} \end{cases} \tag{5}$$

because we have the boundary conditions: $u_x(0, t) = 0 = u_x(l, t)$.
Then the solution is simply given by

$$u(x, t) = \frac{1}{2} \left(\tilde{\phi}(x - ct) + \tilde{\phi}(x + ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} \tilde{\psi}(y) dy. \tag{6}$$

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(ii) If ϕ has singularity at $x_0 \in (0, l)$, then seeing the definition, $\tilde{\phi}$ has singularity at $x_0 - 2kl$ and $2kl - x_0$ for all $k \in \mathbb{Z}$. The solution u is then smooth apart from lines $x + ct = x_0 - 2kl$, $x - ct = x_0 - 2kl$, $x + ct = 2kl - x_0$, $x - ct = 2kl - x_0$ for all $k \in \mathbb{Z}$. Moreover, there is a possibility that the even $2l$−periodic function $\tilde{\phi}$ has singularities at integer multiples of l as well.

Remark. Note that there will be no singularities emanating from these points if all the odd derivatives of ϕ vanish at the endpoints, for then the extension we construct is actually smooth at these points.

Problem 3. Let's solve the inhomogeneous heat equation on the half-line for Dirichlet boundary conditions:

$$u_t - ku_{xx} = f, \quad x \geq 0, \quad u(x, 0) = \phi(x), \quad u(0, t) = 0. \tag{7}$$

(i) Using Duhamel's principle:
We know from the class that the solution of the homogeneous equation on the half line

$$(u_h)_t - k(u_h)_{xx} = 0, \quad x \geq 0, \quad u_h(x, 0) = \phi(x), \quad u_h(0, t) = 0. \tag{8}$$

is

$$\begin{cases} u_h(x, t) = \left(\tilde{S}(t)\phi \right)(x) = \int_{y \geq 0} \tilde{G}(x, y, t)\phi(y)dy \\ \tilde{G}(x, y, t) = (4\pi kt)^{-1/2} \left(e^{-(x-y)^2/(4kt)} - e^{-(x+y)^2/(4kt)} \right). \end{cases} \tag{9}$$

Therefore, Duhamel's principle assures that:

$$\begin{cases} u(x, t) &= \left(\tilde{S}(t)\phi \right)(x) + \int_0^t \left(\tilde{S}(t - s)f_s \right)(x)ds \\ &= \int_{y \geq 0} \tilde{G}(x, y, t)\phi(y)dy + \int_0^t \left(\int_{y \geq 0} \tilde{G}(x, y, t - s)f(s, y)dy \right) ds \\ &= \int_0^{\mp\infty} \tilde{G}(x, y, t)\phi(y)dy + \int_0^t \left(\int_0^{\mp\infty} \tilde{G}(x, y, t - s)f(s, y)dy \right) ds \\ \tilde{G}(x, y, t) &= (4\pi kt)^{-1/2} \left(e^{-(x-y)^2/(4kt)} - e^{-(x+y)^2/(4kt)} \right). \end{cases} \tag{10}$$

(ii) Using extensions:
Since we have the boundary condition $u(0, t) = 0$, we are looking for odd extensions of f and ϕ. That means that we define:

$$f_{odd}(x, t) = \begin{cases} f(x, t), & x \geq 0 \\ -f(-x, t), & x < 0. \end{cases} \tag{11}$$

and

$$\phi_{odd}(x) = \begin{cases} \phi(x), & x \geq 0 \\ -\phi(-x), & x < 0. \end{cases} \tag{12}$$

Now we can look for the solution v of the heat equation on the entire real line:

$$v_t - kv_{xx} = f_{odd}, \quad x \in \mathbb{R}, \quad v(x,0) = \phi_{odd}(x). \tag{13}$$

We know from the course (explicit derivation of Duhamel's principle) that v is given by:

$$\begin{cases} v(x,t) & = (S(t)\phi_{odd})(x) + \displaystyle\int_0^t (S(t-s)(f_{odd})_s)(x)ds \\[2mm] & = \displaystyle\int_{\mathbb{R}} G(x,y,t)\phi_{odd}(y)dy + \int_0^t \left(\int_{\mathbb{R}} G(x,y,t-s)f_{odd}(s,y)dy \right) ds \\[2mm] & = \displaystyle\int_{-\infty}^{+\infty} G(x,y,t)\phi_{odd}(y)dy + \int_0^t \left(\int_{-\infty}^{+\infty} G(x,y,t-s)f_{odd}(s,y)dy \right) ds \\[2mm] G(x,y,t) = (4\pi kt)^{-1/2}e^{-(x-y)^2/(4kt)}. \end{cases} \tag{14}$$

and u is given by $u(x,t) = v(x,t)$, $x \geq 0$. More precisely, we have

$$\begin{aligned} \int_{\mathbb{R}} G(x,y,t)\phi_{odd}(y)dy & = (4\pi kt)^{-1/2} \int_{-\infty}^{+\infty} e^{-(x-y)^2/(4kt)}\phi_{odd}(y)dy \\[2mm] & = (4\pi kt)^{-1/2}\left(-\int_{-\infty}^0 e^{-(x-y)^2/(4kt)}\phi(-y)dy + \int_0^{+\infty} e^{-(x-y)^2/(4kt)}\phi(y)dy \right) \\[2mm] & = (4\pi kt)^{-1/2}\left(-\int_0^{+\infty} e^{-(x+y)^2/(4kt)}\phi(y)dy + \int_0^{+\infty} e^{-(x-y)^2/(4kt)}\phi(y)dy \right) \\[2mm] & = (4\pi kt)^{-1/2} \int_0^{+\infty} \left(e^{-(x-y)^2/(4kt)} - e^{-(x+y)^2/(4kt)} \right)\phi(y)dy \\[2mm] & = \int_0^{+\infty} \tilde{G}(x,y,t)\phi(y)dy \\[2mm] & = \left(\tilde{S}(t)\phi \right)(x) \end{aligned} \tag{15}$$

and in the same fashion,

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}} G(x,y,t-s)f_{odd}(y,s)dyds \\[2mm] & = \int_0^t (4\pi k(t-s))^{-1/2} \int_{-\infty}^{+\infty} e^{-(x-y)^2/(4k(t-s))} f_{odd}(y,s)dyds \\[2mm] & = \int_0^t \left((4\pi k(t-s))^{-1/2}\left(-\int_{-\infty}^0 e^{-(x-y)^2/(4k(t-s))} f(-y,s)dy + \int_0^{+\infty} e^{-(x-y)^2/(4k(t-s))} f(y,s)dy \right) \right) ds \\[2mm] & = \int_0^t \left((4\pi k(t-s))^{-1/2}\left(-\int_0^{+\infty} e^{-(x+y)^2/(4k(t-s))} f(y,s)dy + \int_0^{+\infty} e^{-(x-y)^2/(4k(t-s))} f(y,s)dy \right) \right) ds \\[2mm] & = \int_0^t \left((4\pi k(t-s))^{-1/2} \int_0^{+\infty} \left(e^{-(x-y)^2/(4k(t-s))} - e^{-(x-y)^2/(4k(t-s))} \right) f(y,s)dy \right) ds \\[2mm] & = \int_0^t \int_0^{+\infty} \tilde{G}(x,y,t-s)f(y,s)dyds = \int_0^t \left(\tilde{S}(t-s)f_s \right)(x)ds. \end{aligned} \tag{16}$$

Therefore we have indeed obtained the same solution as in (a).

Problem 4. We transform the wave equation on $\mathbb{R}$

$$u_{tt} - c^2 u_{xx} = f, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \tag{17}$$

into the first order PDE system:

$$\begin{cases} \partial_t U - AU = \begin{bmatrix} 0 \\ f \end{bmatrix}, \quad U(x,0) = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}, \\ A = \begin{bmatrix} 0 & Id \\ c^2 \partial_x^2 & 0 \end{bmatrix}. \end{cases} \tag{18}$$

Then using Duhamel's principle, we already know that the solution is going to be:

$$U(x,t) = \mathcal{S}(t) \begin{bmatrix} \phi \\ \psi \end{bmatrix}(x) + \int_0^t \mathcal{S}(t-s) \begin{bmatrix} 0 \\ f_s \end{bmatrix}(x)ds. \tag{19}$$

The only remaining bit of work is to define what $\mathcal{S}(t)$ is. Let's give 3 ways of getting it.

The direct method. We already know the solution of the homogeneous wave equation, namely:

$$u_h(x,t) = \frac{1}{2}\left(\phi(x-ct) + \phi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds. \tag{20}$$

Therefore, we can take its derivative with respect to time, and get

$$v_h(x,t) = (u_h)_t(x,t) = \frac{c}{2}\left(\phi'(x+ct) - \phi'(x-ct)\right) + \frac{1}{2}\left(\psi(x-ct) + \psi(x+ct)\right), \tag{21}$$

hence we get

$$\begin{aligned} U_h(x,t) \quad &= \mathcal{S}(t) \begin{bmatrix} \phi \\ \psi \end{bmatrix}(x) \\ &= \begin{bmatrix} \frac{1}{2}\left(\phi(x-ct) + \phi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds \\ \frac{c}{2}\left(\phi'(x+ct) - \phi'(x-ct)\right) + \frac{1}{2}\left(\psi(x-ct) + \psi(x+ct)\right) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}\left(\phi(x-ct) + \phi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds \\ \frac{c}{2}\left(\phi'(x+ct) - \phi'(x-ct)\right) + \frac{1}{2}\left(\psi(x-ct) + \psi(x+ct)\right) \end{bmatrix} \end{aligned} \tag{22}$$

Therefore, if we insist on writing explicitly $\mathcal{S}(t)$, we can choose:

$$\begin{cases} \mathcal{S}(t) = \frac{1}{2c} \begin{bmatrix} D\left(\mathcal{T}_{-ct} - \mathcal{T}_{ct}\right) & \mathcal{T}_{-ct} - \mathcal{T}_{ct} \\ D^2\left(\mathcal{T}_{-ct} - \mathcal{T}_{ct}\right) & D\left(\mathcal{T}_{-ct} - \mathcal{T}_{ct}\right) \end{bmatrix} \\ \left(\mathcal{T}_\alpha h\right)(x) = \int_0^{x-\alpha} h(s)ds, \quad (Dh)(x) = h'(x). \end{cases} \tag{23}$$

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And the solution u writes:

$$
\begin{aligned}
u(x,t) &= (\mathcal{S}(t)U)_1(x) + \int_0^t \left(\mathcal{S}(t-s) \begin{bmatrix} 0 \\ f_s \end{bmatrix} \right)_1 (x) ds \\
&= \frac{1}{2c} \left((D\left(\mathcal{T}_{-ct} - \mathcal{T}_{ct}\right)\phi)(x) + ((\mathcal{T}_{-ct} - \mathcal{T}_{ct})\psi)(x) + \int_0^t \left((\mathcal{T}_{-c(t-s)} - \mathcal{T}_{c(t-s)}\right)f_s)(x)ds \right) \\
&= \frac{1}{2c} \left(c\left(\phi(x+ct) + \phi(x-ct)\right) + \int_{x-ct}^{x+ct} \psi(y)dy + \int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy \right) ds \right) \\
&= \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy + \frac{1}{2c}\int_0^t \left(\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy \right) ds.
\end{aligned}
\tag{24}
$$

Fourier transform & eigenvalue decomposition.   Taking the partial Fourier transform in x in the PDE system gives:

$$
\begin{cases}
\partial_t \hat{U} - \hat{A}\hat{U} = \begin{bmatrix} 0 \\ \hat{f} \end{bmatrix}, & \hat{U}(\xi,0) = \begin{bmatrix} \hat{\phi}(\xi) \\ \hat{\psi}(\xi) \end{bmatrix}, \\
\hat{A} = \begin{bmatrix} 0 & 1 \\ -c^2\xi^2 & 0 \end{bmatrix}.
\end{cases}
\tag{25}
$$

We can then directly write the solution of the ODE:

$$
\begin{cases}
\hat{U}(\xi,t) = e^{t\hat{A}}\hat{U}(\xi,0), \\
\hat{A} = \begin{bmatrix} 0 & 1 \\ -c^2\xi^2 & 0 \end{bmatrix}.
\end{cases}
\tag{26}
$$

Now we are left with computing $e^{t\hat{A}}$ before finishing by taking the inverse Fourier transform.

Here we choose to diagonalize $\hat{A}$. Indeed, by looking at the trace, the determinant of $\hat{A}$ and with a bit of algebra, we get that $ic\xi$ and $-ic\xi$ associated to the respective eigenvectors $\begin{bmatrix} 1 \\ ic\xi \end{bmatrix}$ and $\begin{bmatrix} -1 \\ ic\xi \end{bmatrix}$. Therefore, we can write

$$
\begin{cases}
\hat{A} = \Omega \begin{bmatrix} ic\xi & 0 \\ 0 & -ic\xi \end{bmatrix} \Omega^{-1} \\
\Omega = \begin{bmatrix} 1 & -1 \\ ic\xi & ic\xi \end{bmatrix}, \quad \Omega^{-1} = \frac{1}{2ic\xi} \begin{bmatrix} ic\xi & 1 \\ -ic\xi & 1 \end{bmatrix}
\end{cases}
\tag{27}
$$

Now $e^{t\hat{A}}$ becomes

$$
\begin{aligned}
e^{t\hat{A}} &= \Omega \begin{bmatrix} e^{ict\xi} & 0 \\ 0 & e^{-ict\xi} \end{bmatrix} \Omega^{-1} \\
&= \begin{bmatrix} \dfrac{e^{ict\xi} + e^{-ict\xi}}{2} & \dfrac{1}{ic\xi}\dfrac{e^{ict\xi} - e^{-ict\xi}}{2} \\ ic\xi\dfrac{e^{ict\xi} - e^{-ict\xi}}{2} & \dfrac{e^{ict\xi} + e^{-ict\xi}}{2} \end{bmatrix}
\end{aligned}
\tag{28}
$$

And finally, using HW5 and lecture notes on the Fourier transform, we can formally write

$$
\mathcal{S}(t) = \mathcal{F}_\xi^{-1} e^{t\hat{A}} \quad = \mathcal{F}_\xi^{-1} \left[\begin{array}{cc} \dfrac{e^{ict\xi} + e^{-ict\xi}}{2} & \dfrac{1}{ic\xi}\dfrac{e^{ict\xi} - e^{-ict\xi}}{2} \\ ic\xi\dfrac{e^{ict\xi} - e^{-ict\xi}}{2} & \dfrac{e^{ict\xi} + e^{-ict\xi}}{2} \end{array} \right] \tag{29}
$$

$$
= \frac{1}{2c} \left[\begin{array}{cc} D\left(\mathcal{T}_{-ct} - \mathcal{T}_{ct}\right) & \mathcal{T}_{-ct} - \mathcal{T}_{ct} \\ D^2\left(\mathcal{T}_{-ct} - \mathcal{T}_{ct}\right) & D\left(\mathcal{T}_{-ct} - \mathcal{T}_{ct}\right) \end{array} \right]
$$

Fourier transform & direct calculation. Another way of dealing with finding $\mathcal{S}(t)$ is to do the same thing as previously, but to calculate $e^{t\hat{A}}$, use its definition:

$$
e^{t\hat{A}} = \sum_{k=0}^{+\infty} \frac{t^k \hat{A}^k}{k!}. \tag{30}
$$

Now let's have a look at $\hat{A}^2$:

$$
\begin{aligned}
\hat{A}^2 &= \left[\begin{array}{cc} 0 & 1 \\ -c^2\xi^2 & 0 \end{array} \right] \left[\begin{array}{cc} 0 & 1 \\ -c^2\xi^2 & 0 \end{array} \right] \\
&= \left[\begin{array}{cc} -c^2\xi^2 & 0 \\ 0 & -c^2\xi^2 \end{array} \right] \\
&= -c^2\xi^2 Id,
\end{aligned} \tag{31}
$$

and by direct recursion,

$$
\left\{ \begin{array}{l} \hat{A}^{2k} = (-1)^k (c\xi)^{2k} Id, \\ \hat{A}^{2k+1} = (-1)^k (c\xi)^{2k} \hat{A} = \left[\begin{array}{cc} 0 & (-1)^k(c\xi)^{2k} \\ -(-1)^k(c\xi)^{2k+2} & 0 \end{array} \right] \end{array} \right., \forall k \in \mathbb{N}. \tag{32}
$$

Therefore

$$
\begin{aligned}
e^{t\hat{A}} &= \left[\begin{array}{cc} \displaystyle\sum_{k=0}^{+\infty} \frac{(-1)^k (ct\xi)^{2k}}{(2k)!} & \displaystyle\frac{1}{c\xi}\sum_{k=0}^{+\infty} \frac{(-1)^k (ct\xi)^{2k+1}}{(2k+1)!} \\ -c\xi\displaystyle\sum_{k=0}^{+\infty} \frac{(-1)^k (ct\xi)^{2k+1}}{(2k+1)!} & \displaystyle\sum_{k=0}^{+\infty} \frac{(-1)^k (ct\xi)^{2k}}{(2k)!} \end{array} \right] \\
&= \left[\begin{array}{cc} \cos(ct\xi) & \dfrac{1}{c\xi}\sin(ct\xi) \\ -c\xi\sin(ct\xi) & \cos(ct\xi) \end{array} \right] \\
&= \left[\begin{array}{cc} \dfrac{e^{ict\xi} + e^{-ict\xi}}{2} & \dfrac{1}{ic\xi}\dfrac{e^{ict\xi} - e^{-ict\xi}}{2} \\ ic\xi\dfrac{e^{ict\xi} - e^{-ict\xi}}{2} & \dfrac{e^{ict\xi} + e^{-ict\xi}}{2} \end{array} \right].
\end{aligned} \tag{33}
$$

And we get the same result (of course!).

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Problem 5.

(i) We are looking for the solutions of the eigenvalue problem on $[0, l]$:

$$-X'' = \lambda X, \; X(0) = 0, \; X'(l) = 0. \tag{34}$$

For $\lambda < 0$, the solution writes $X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$ with $A, B \in \mathbb{R}$. Since $X(0) = 0$, we have $A + B = 0$, meaning that $X(x) = 2A\sinh(\sqrt{-\lambda}x)$. Now with $X'(l) = 0$, we get $A = 0$ and $X \equiv 0$, which means that there is no strictly negative eigenvalue.
For $\lambda = 0$, the solution writes $X(x) = Ax + B$. Since $X(0) = 0$, we have $B = 0$, meaning that $X(x) = Ax$. Now with $X'(l) = 0$, we get $A = 0$ and $X \equiv 0$, which means that 0 is not an eigenvalue.
Finally, for $\lambda > 0$, the solution writes $X(x) = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{-\lambda}x)$. Since $X(0) = 0$, we have $A = 0$, meaning that $X(x) = B\sin(\sqrt{\lambda}x)$. Now with $X'(l) = 0$, we get that

$$\sqrt{\lambda}l = \frac{(2n+1)\pi}{2}, \; n \in \mathbb{Z}. \tag{35}$$

That means that we have a countable number of eigenvalues (indexed by $n \in \mathbb{Z}$), and we can write

$$\lambda_n = \left(\frac{(2n+1)\pi}{2l} \right)^2, \; n \in \mathbb{Z}, \tag{36}$$

and corresponding eigenfunctions are

$$X_n(x) = \sin(\sqrt{\lambda_n}x) = \sin\left(\frac{(2n+1)}{2l}\pi x \right), \; n \in \mathbb{Z}. \tag{37}$$

(ii) Let us consider the following wave equation

$$u_{tt} = c^2 u_{xx}, \; u(0, t) = 0, \; u_x(l, t) = 0. \tag{38}$$

We are looking for separated solutions of the form $u(x, t) = X(x)T(t)$. Assuming that X and T do not vanish, the PDE gives

$$\frac{T''(t)}{T(t)} = c^2 \frac{X''(x)}{X(x)}, \; X(0) = 0, \; X'(l) = 0. \tag{39}$$

Therefore $\frac{X''(x)}{X(x)}$ is a (true) constant, say $\frac{X''(x)}{X(x)} = -\lambda$, with $\lambda \in \mathbb{R}$. But from (i), we know that the possible values for λ are λ_n (and we have the associated eigenfunctions X_n). We are then left with determining $T(t)$, but we have for λ_n the following ODE:

$$T_n''(t) = -c^2 \lambda_n T_n(t), \tag{40}$$

which gives the general solution

$$T_n(t) = A_n \cos(c\sqrt{\lambda_n}t) + B_n \sin(c\sqrt{\lambda_n}t)$$
$$= A_n \cos\left(\frac{(2n+1)}{2l}c\pi t\right) + B_n \sin\left(\frac{(2n+1)}{2l}c\pi t\right), \ n \in \mathbb{Z}. \tag{41}$$

and the general separated solution is

$$u(x,t) = \sum_{n \in \mathbb{Z}} \sin\left(\frac{(2n+1)}{2l}\pi x\right) \left(A_n \cos\left(\frac{(2n+1)}{2l}c\pi t\right) + B_n \sin\left(\frac{(2n+1)}{2l}c\pi t\right)\right). \tag{42}$$

(iii) $u_t(x,0) = 0$ gives

$$u_t(x,0) = \sum_{n \in \mathbb{Z}} A_n \sqrt{\lambda_n} \sin\left(\frac{(2n+1)}{2l}\pi x\right) = \sum_{n \in \mathbb{Z}} A_n \sqrt{\lambda_n} X_n(x) = 0. \tag{43}$$

Therefore, by orthogonality of the X_n's, we get that $A_n = 0$, for all $n \in \mathbb{Z}$. The other initial condition gives

$$u(x,0) = \sum_{n \in \mathbb{Z}} B_n \sin\left(\frac{(2n+1)}{2l}\pi x\right) = \sum_{n \in \mathbb{Z}} B_n X_n(x) = X_1(x) - 2X_2(x). \tag{44}$$

Hence, again by orthogonality, we deduce that $A_1 = 1$, $A_2 = -2$ and $A_n = 0$ otherwise.
The solution is then

$$u(x,t) = \sin\left(\frac{3\pi x}{2l}\right) \cos\left(\frac{3c\pi t}{2l}\right) - 2\sin\left(\frac{5\pi x}{2l}\right) \cos\left(\frac{5c\pi t}{2l}\right) \tag{45}$$

(iv) The reasoning is the same as in (ii).
Let us consider the following heat equation

$$u_t = ku_{xx}, \ u(0,t) = 0, \ u_x(l,t) = 0, \tag{46}$$

with $k > 0$.
We are looking for separated solutions of the form $u(x,t) = X(x)T(t)$. Assuming that X and T do not vanish, the PDE gives

$$\frac{T'(t)}{T(t)} = k\frac{X''(x)}{X(x)}, \ X(0) = 0, \ X'(l) = 0. \tag{47}$$

Therefore $\frac{X''(x)}{X(x)}$ is a (true) constant, say $\frac{X''(x)}{X(x)} = -\lambda$, with $\lambda \in \mathbb{R}$. But from (i), we know that the possible values for λ are λ_n (and we have the associated

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eigenfunctions X_n). We are then left with determining $T(t)$, but we have for λ_n the following ODE:

$$T_n'(t) = -k\lambda_n T_n(t), \tag{48}$$

which gives the general solution

$$T_n(t) = A_n e^{-k\lambda_n t} = A_n e^{-k\left(\frac{(2n+1)\pi}{2l}\right)^2 t} \tag{49}$$

and the general separated solution is

$$u(x,t) = \sum_{n\in\mathbb{Z}} A_n e^{-k\left(\frac{(2n+1)\pi}{2l}\right)^2 t} \sin\left(\frac{(2n+1)}{2l}\pi x\right). \tag{50}$$

Problem 6.

(i) The eigenvalue/eigenfunction problem we have to solve reduces to:

$$X''(x) = -\lambda X(x), \ X(l) = X(-l), \ X'(l) = X'(-l). \tag{51}$$

For $\lambda = 0$, we have $\lambda_0 = 0$, and $X_0(x) = 1$.

For $\lambda > 0$, we have, after solving the ODE, $\lambda_n = \left(\frac{n\pi}{l}\right)^2$, and

$$X_n(x) = A_n \cos\left(\frac{n\pi x}{l}\right) + B_n \sin\left(\frac{n\pi x}{l}\right), \ n \geq 1. \tag{52}$$

For $\lambda < 0$, we get $X(x) = 0$. Therefore there is no strictly negative eigenvalue.

Then we can solve for T_n for each value of λ_n. We have the ODE

$$T_n''(t) = -c^2\lambda_n T_n(t), \tag{53}$$

which gives the solutions:

$$\begin{aligned} T_0(t) &= C_0 + D_0 t, \\ T_n(t) &= C_n \cos\left(\frac{nc\pi t}{l}\right) + D_n \sin\left(\frac{nc\pi t}{l}\right), \ n \geq 1. \end{aligned} \tag{54}$$

Therefore the general separated solution is which gives the solutions:

$$\begin{aligned} u(x,t) &= A_0 + D_0 t + \sum_{n=1}^{+\infty} A_n \cos\left(\frac{nc\pi t}{l}\right)\cos\left(\frac{n\pi x}{l}\right) \\ &+ \sum_{n=1}^{+\infty} B_n \cos\left(\frac{nc\pi t}{l}\right)\sin\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{+\infty} C_n \sin\left(\frac{nc\pi t}{l}\right)\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{+\infty} D_n \sin\left(\frac{nc\pi t}{l}\right)\sin\left(\frac{n\pi x}{l}\right). \end{aligned}$$
$$\tag{55}$$

(with arbitrary A_n, B_n, C_n and D_n.)

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(ii) Initial condition $u(x,0) = 0$ gives $A_n = B_n = 0$ for all $n \in \mathbb{N}$.
Initial condition $u_t(x,0) = \cos\left(\dfrac{2\pi x}{l}\right) - \sin\left(\dfrac{\pi x}{l}\right)$ implies that

$D_0 = 0$, $D_1 = -\dfrac{l}{c\pi}$, $C_2 = \dfrac{1}{2c\pi}$ and $C_n = D_n = 0$ for the other n's.
Hence the solution is

$$u(x,t) = \frac{l}{2c\pi} \sin\left(\frac{2c\pi t}{l}\right) \cos\left(\frac{2\pi x}{l}\right) - \frac{l}{c\pi} \sin\left(\frac{c\pi t}{l}\right) \sin\left(\frac{\pi x}{l}\right). \tag{56}$$

(iii) Consider $\tilde{u}$, the $2l$-periodic extension of u. Then $\tilde{u}$ verifies the following PDE:

$$\tilde{u}_{tt} - c^2 \tilde{u}_{xx} = 0, \ \ \tilde{u}(x,0) = 0, \ \ \tilde{u}_t(x,0) = \cos\left(\frac{2\pi x}{l}\right) - \sin\left(\frac{\pi x}{l}\right). \tag{57}$$

By d'Alembert's formula, we know that

$$
\begin{aligned}
\tilde{u}(x,t) &= \frac{1}{2c} \int_{x-ct}^{x+ct} \left(\cos\left(\frac{2\pi y}{l}\right) - \sin\left(\frac{\pi y}{l}\right) \right) dy \\
&= \frac{l}{4c\pi} \sin\left(\frac{2\pi y}{l}\right)\Big|_{x-ct}^{x+ct} + \frac{l}{2c\pi} \cos\left(\frac{\pi y}{l}\right)\Big|_{x-ct}^{x+ct} \\
&= \frac{l}{4c\pi} \left(\sin\left(\frac{2\pi(x+ct)}{l}\right) - \sin\left(\frac{2\pi(x-ct)}{l}\right) \right) + \frac{l}{2c\pi} \left(\cos\left(\frac{\pi(x+ct)}{l}\right) - \cos\left(\frac{\pi(x-ct)}{l}\right) \right) \\
&= \frac{l}{2c\pi} \sin\left(\frac{2c\pi t}{l}\right) \cos\left(\frac{2\pi x}{l}\right) - \frac{l}{c\pi} \sin\left(\frac{c\pi t}{l}\right) \sin\left(\frac{\pi x}{l}\right).
\end{aligned}
\tag{58}
$$

(iv) u will have singularities on lines $x \pm ct = x_0 + 2kl$, where $k \in \mathbb{Z}$. Consider the strip to be a circle. In other words, the head and the tail are connected. The singularity at the beginning will propagate at speed c in both directions. At time t, it has run a distance of ct, so it has reached $x_0 \pm ct$. u has singularities at $(x_0 \pm ct \mod 2l, t)$, where $y \mod 2l$ means the unique point $z \in [-l, l)$ such that $y - z$ is a multiple of $2l$. In other words, singularities will keep traveling around the ring at speed c. And since the ring has perimeter $2l$, after traveling for time $\dfrac{2l}{c}$, or any integer multiple of this, they are back to where they started. From the perspective of the method of images (going to the reals, i.e. the universal cover), one has a singularity in the initial data at x_0 plus integer multiples of $2l$; these propagate in the standard manner for the wave equation on the real line (at speed c); if one takes $[-l, l]$ as the interval representing the circle, when one of these is in this interval (at some time), it gives rise to a singularity you observe.
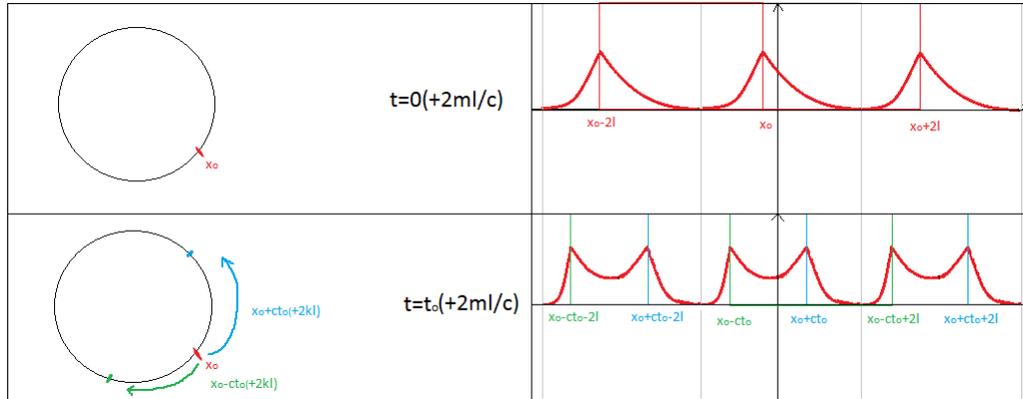Here is a sketch of what is going on:

Figure 1: Left: circular representation (perimeter is $2l$). Right: representation on the real line ($2l$-periodic function). $m \in \mathbb{Z}$, $k \in \mathbb{Z}$, $-l \leq x_0 \leq l$