Problem 1.  
(i) Using the general ‘separated’ solution you found in Problem Set 7, Problem 3, solve the wave equation 
\[ u_{tt} = c^2 u_{xx}, \quad u(0, t) = 0, \quad u_x(\ell, t) = 0, \]
with initial conditions 
\[ u(x, 0) = 0, \quad u_t(x, 0) = x(x - \ell)^2. \]
(ii) Using the general ‘separated’ solution you found in Problem Set 7, Problem 3, solve the heat equation 
\[ u_t = ku_{xx}, \quad u(0, t) = 0, \quad u(\ell, t) = 0, \quad u(x, 0) = x(x - \ell)^2. \]
You may assume throughout that the generalized Fourier series you construct converge to the function they are supposed to represent.

Problem 2. Let \( \phi(x) = |x| \) on \([-\pi, \pi]\). Let 
\[ f(x) = a_0 + a_1 \cos x + a_2 \cos(2x) + b_1 \sin x + b_2 \sin(2x), \]
with what choice of the coefficients \( a_j \) and \( b_j \) is the \( L^2 \) error \( \|f - \phi\| \) minimal? (Here \( \|f - \phi\|^2 = \int_{-\pi}^{\pi} |f(x) - \phi(x)|^2 \, dx \).)

Problem 3. Let \( V = C([0, \ell]) \) with the inner product \( \langle f, g \rangle = \int_0^\ell f(x) g(x) \, dx \). If \( D \subset V \) is a subspace, and \( A : D \to V \) is symmetric, we say that \( A \) is positive if \( \langle Av, v \rangle \geq 0 \) for all \( v \in D \).
(i) Show that if \( A \) is positive than all eigenvalues \( \lambda \) of \( A \) are \( \geq 0 \).
(ii) Show that \( A = -\frac{d^2}{dx^2} \) with domain 
\[ D = \{ f \in C^2([0, \ell]) : f(0) = f'(\ell) = 0 \} \]
is symmetric and is positive.
(iii) Show that \( A = \frac{d^4}{dx^4} \) with domain 
\[ D = \{ f \in C^4([0, \ell]) : f(0) = f'(0) = f(\ell) = f'(\ell) = 0 \} \]
is symmetric and is positive.

Problem 4. Let \( \gamma_n \) be a sequence of constants tending to \( \infty \). Consider the following continuous functions \( f_n, n \geq 2, \) on \([0, 1]\):
\[ f_n(x) = \begin{cases} 
2n\gamma_n x, & 0 \leq x \leq \frac{1}{2n}, \\
2n\gamma_n \left( \frac{1}{n} - x \right), & \frac{1}{2n} \leq x \leq \frac{1}{n}, \\
0, & \frac{1}{n} \leq x \leq 1.
\end{cases} \]
(i) Sketch the graph of a few \( f_n \).
(ii) Show that \( f_n \to 0 \) pointwise.
(iii) Show that the convergence is not uniform.
(iv) Show that \( f_n \to 0 \) in \( L^2 \) if \( \gamma_n = n^{1/4} \).
(v) Show that \( f_n \) does not converge in \( L^2 \) if \( \gamma_n = n \).