CME 303/MATH 220: PROBLEM SET 8, PART II
DO NOT HAND IN!

Problem 1. Let $\phi(x) = x(\ell - x)$ on $[0, \ell]$.

(i) Find the Fourier sine series of ϕ, and state what it converges to for $x \in \mathbb{R}$.
(ii) Find the Fourier cosine series of ϕ, and state what it converges to for $x \in \mathbb{R}$.
(iii) Compare the decay rates of the coefficients of the two series as $n \to \infty$. Why do the coeffients decay faster in one of the cases?

Problem 2. For both of the following functions discuss whether the Fourier sine series converges uniformly or in L^2:

(i) $\phi(x) = x$ on $[0, \ell]$,
(ii) $\phi(x) = x(\ell - x)^2$ on $[0, \ell]$,

Justify your answer by quoting the relevant convergence theorems. You do not need to compute the respective Fourier series.

Problem 3. In this problem we consider the inhomogeneous wave equation on $[0, \ell]$ with inhomogeneous Dirichlet boundary conditions:

$$u_{tt} - c^2 u_{xx} = f,\ u(0,t) = h(t),\ u(\ell,t) = j(t),\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x).$$

(i) Suppose $h(t) = 0 = j(t)$ for all t. For each t, we can expand $u(x,t)$ (which is a function of x only then) in Fourier sine series in x, with coefficients depending on t:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin(n\pi x/\ell),$$

and can obtain a similar expansion for f.

(ii) Assuming that you can differentiate term by term, i.e. that the odd 2ℓ-periodic extension of u is well-behaved, find the Fourier sine series of u_{xx} and u_{tt}.
(iii) Derive an ODE for $u_n(t)$ by substituting the Fourier series into the PDE. Find the initial conditions satisfied by $u_n(t)$, and solve the ODE.
(iv) Now do not assume that $h(t) = 0 = j(t)$. We could solve the problem by subtracting from u a function F that satisfies the boundary conditions to place ourselves into the previous scenario. Instead, we proceed as follows.

As u does *not* satisfy the homogeneous boundary conditions that sine does, the Fourier coefficients will decay slowly, and term by term differentiation is not allowed to calculate u_{xx}. Instead, let $w(x,t) = u_{xx}(x,t)$, and expand w in Fourier sine series:

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t) \sin(n\pi x/\ell).$$

Calculate the coefficients $w_n(t)$ in terms of w, and then integrate by parts twice to obtain them in terms of $u_n(t)$, $h(t)$ and $j(t)$.

(v) Obtain an ODE for $u_n(t)$, and solve it.