MATH 220: Problem Set 8b
Solutions

Problem 1.

(i) Let the Fourier sine series of $\phi$ be $\sum_{n=1}^{\infty} A_n \sin \left(\frac{n\pi x}{l}\right)$. We can compute the coefficients as:

$$A_n = \frac{2}{l} \int_{l}^{0} \phi(x) \sin \left(\frac{n\pi x}{l}\right) \, dx = \frac{4(1 - (-1)^n)}{l} \left(\frac{l}{n\pi}\right)^3.$$  \hspace{1cm} (1)

Now we know from the lecture notes about convergence of Fourier series that the sine Fourier series converges to an odd 2$l$-periodic extension function of $\phi$.

(ii) Similarly, let the Fourier cosine series of $\phi$ be $\sum_{n=1}^{\infty} B_n \cos \left(\frac{n\pi x}{l}\right)$. We can compute the coefficients as:

$$B_0 = \frac{1}{l} \int_{0}^{l} \phi(x) \, dx = \frac{l^3}{6}, \quad B_n = \frac{2}{l} \int_{0}^{l} \phi(x) \cos \left(\frac{n\pi x}{l}\right) \, dx = ((1 + (-1)^n) \frac{l^3}{n^3\pi^3}, \quad n \geq 1.$$ \hspace{1cm} (2)

Now we know from the lecture notes about convergence of Fourier series that the cosine Fourier series converges to an even 2$l$-periodic extension function of $\phi$.

(iii) We can see that $A_n$ decays as $\frac{1}{n^3}$ while $A_n$ decays as $\frac{1}{n^2}$, meaning that the former one is faster. The reason is that the odd extension of $\phi$ is continuously differentiable while the even one is merely continuous. And we know that the smoother the function is, the faster the Fourier coefficients decay.

Problem 2. For Fourier sine series, we first make an odd extension of $\phi$. Now our $\phi$ is defined on $[-l, l]$.
Fourier sine series converge pointwisely to \( \phi \) in \((-l, l)\) and to \( \frac{\phi(l) + \phi(-l)}{2} = 0 \) at \( x = \pm l \).

However, we have \( \phi(-l) = -l \neq l = \phi(l) \). Therefore the convergence is not uniform (otherwise the limit function should be continuous).

Now, we obviously have \( \int_{-l}^{l} \phi(x)^2 \, dx < +\infty \). This implies that \( \phi \in L^2([-l, l]) \).

Therefore we know from the course that the Fourier sine series converge to \( \phi \) in \( L^2 \).

Using similar arguments, Fourier sine series converge pointwisely to \( \phi \) in \((-l, l)\) and to \( \frac{\phi(l) + \phi(-l)}{2} = 0 \) at \( x = \pm l \).

Moreover, here we have \( \phi(l) = \phi(-l) = 0 \) and \( \phi'(l) = \phi'(-l) = 0 \). Therefore the extension of \( \phi \) is \( C^2 \), and the Fourier sine series converges uniformly to \( \phi \) (lecture notes).

What’s more, we obviously have \( \int_{-l}^{l} \phi(x)^2 \, dx < +\infty \). This implies that \( \phi \in L^2([-l, l]) \). Therefore we know from the course that the Fourier sine series converge to \( \phi \) in \( L^2 \). Or we could have simply said that uniform convergence implies convergence in \( L^2 \).

**Problem 3.**

There is no question. We know from the course that we have (at least in the \( L^2 \) sense) the expansions

\[
\begin{align*}
\phi(x) &= \sum_{n=1}^{+\infty} \phi_n \sin \left( \frac{n \pi x}{l} \right), \\
\psi(x) &= \sum_{n=1}^{+\infty} \psi_n \sin \left( \frac{n \pi x}{l} \right). \\
\end{align*}
\]

\[ u(x, t) = \sum_{n=1}^{+\infty} u_n(t) \sin \left( \frac{n \pi x}{l} \right), \]

\[ f(x, t) = \sum_{n=1}^{+\infty} f_n(t) \sin \left( \frac{n \pi x}{l} \right), \]

\[ \phi(x) = \sum_{n=1}^{+\infty} \phi_n \sin \left( \frac{n \pi x}{l} \right), \]

\[ \psi(x) = \sum_{n=1}^{+\infty} \psi_n \sin \left( \frac{n \pi x}{l} \right). \]
(ii) If we assume that we can differentiate term by term the Fourier sine series of \( u \), we get:

\[
\begin{align*}
    u_{tt}(x,t) &= \sum_{n=1}^{+\infty} u''_n(t) \sin \left( \frac{n\pi x}{l} \right), \\
    u_{xx}(x,t) &= \sum_{n=1}^{+\infty} \left( -\frac{n^2\pi^2}{l^2} u_n(t) \right) \sin \left( \frac{n\pi x}{l} \right). 
\end{align*}
\] (4)

(iii) By using orthogonality (or identifying coefficients term by term), we get the ODE:

\[
\begin{align*}
    \left\{ \begin{array}{l}
    u''_n(t) + \left( \frac{n^2\pi^2}{l^2} u_n(t) \right) = f_n(t), \\
    u_n(0) = \phi_n, \quad u'_n(0) = \psi_n.
    \end{array} \right. 
\] (5)

Now using Duhamel’s principle, (or the method of variation of the parameters), we get the solution

\[
    u_n(t) = \psi_n \frac{l}{cn\pi} \sin \left( \frac{cn\pi t}{l} \right) + \phi_n \cos \left( \frac{cn\pi t}{l} \right) + \int_0^t \frac{l}{cn\pi} \sin \left( \frac{cn\pi(t-s)}{l} \right) f_n(s) ds. 
\] (6)

(iv) We have

\[
    w(x,t) = \sum_{n=1}^{+\infty} w_n(t) \sin \left( \frac{n\pi x}{l} \right), 
\] (7)

and by integration by parts, we get that

\[
\begin{align*}
    w_n(t) &= \frac{2}{l} \int_0^l u_{xx}(x,t) \sin \left( \frac{n\pi x}{l} \right) dx \\
    &= -\frac{2n\pi}{l^2} \int_0^l u_x(x,t) \cos \left( \frac{n\pi x}{l} \right) dx \\
    &= -\frac{2n\pi}{l^2} \left( u_x(x,t) \cos \left( \frac{n\pi x}{l} \right) \bigg|_{x=0}^{x=l} + \frac{n\pi}{l} \int_0^l u(x,t) \sin \left( \frac{n\pi x}{l} \right) dx \right) \\
    &= -\frac{2n\pi}{l^2} \left( (-1)^n j(t) - h(t) \right) - \frac{n^2\pi^2}{l^2} u_n(t). 
\end{align*}
\] (8)
In the same fashion as previously, identifying the coefficients in the PDE, we get:

\[ u''_n(t) = c^2 w_n(t) + f_n(t) = f_n(t) - \frac{2n c^2 \pi}{l^2} ((-1)^n j(t) - h(t)) - \frac{n^2 c^2 \pi^2}{l^2} u_n(t). \] (9)

Therefore, the ODE to solve is now:

\[
\begin{aligned}
& \left\{
\begin{array}{l}
  u''_n(t) + \left(\frac{n^2 c^2 \pi^2}{l^2} u_n(t)\right) = f_n(t) - \frac{2n c^2 \pi}{l^2} ((-1)^n j(t) - h(t)) - \frac{n^2 c^2 \pi^2}{l^2} u_n(t), \\
  u_n(0) = \phi_n, \quad u'_n(0) = \psi_n.
\end{array}
\right.
\] (10)

Calling \( g_n(t) = f_n(t) - \frac{2n c^2 \pi}{l^2} ((-1)^n j(t) - h(t)) - \frac{n^2 c^2 \pi^2}{l^2} u_n(t), \) and using Duhamel’s principle, we finally get the solution

\[ u_n(t) = \psi_n \frac{l}{cn\pi} \sin\left(\frac{cn\pi t}{l}\right) + \phi_n \cos\left(\frac{cn\pi t}{l}\right) + \int_0^t \frac{l}{cn\pi} \sin\left(\frac{cn\pi (t-s)}{l}\right) g_n(s) ds. \] (11)