MATH 220: Problem Set 8
Solutions

Problem 1.

(i) We know from PSET 7, that the general separated solution writes

$$u(x,t) = \sum_{n\in\mathbb{Z}} \sin\left(\frac{(2n+1)}{2l}\pi x\right)\left(A_n \cos\left(\frac{(2n+1)}{2l}c\pi t\right) + B_n \sin\left(\frac{(2n+1)}{2l}c\pi t\right)\right). \tag{1}$$

Using the same reasoning as before, the initial conditions gives

$$\begin{cases} A_n = 0, \ \forall n \in \mathbb{Z}, \\ x(x-l)^2 = \sum_{n\in\mathbb{Z}} \left(\frac{2n+1}{2l}c\pi\right) B_n \sin\left(\frac{(2n+1)}{2l}\pi x\right). \end{cases} \tag{2}$$

Therefore, if we call $f(x) = x(x-l)^2$, we have

$$\begin{aligned} B_n &= \left(\frac{2lc\pi}{2n+1}\right)\frac{\langle f, X_n\rangle}{\|X_n\|^2} \\ &= \left(\frac{2lc\pi}{2n+1}\right)\frac{2}{l}\int_0^l x(x-l)^2 \sin\left(\frac{(2n+1)}{2l}\pi x\right) dx \\ &= 4\left((2n+1)\pi - 3(-1)^n\right)\frac{l^4}{c\left(\left(\frac{2n+1}{2}\right)\pi\right)^5}. \end{aligned} \tag{3}$$

(ii) We know from PSET 7 that the general separated solution writes

$$u(x,t) = \sum_{n\in\mathbb{Z}} A_n e^{-k\left(\frac{(2n+1)\pi}{2l}\right)^2 t} \sin\left(\frac{(2n+1)}{2l}\pi x\right). \tag{4}$$

Using the same reasoning as before, the initial conditions gives

$$\left\{ x(x-l)^2 = \sum_{n\in\mathbb{Z}} A_n \sin\left(\frac{(2n+1)}{2l}\pi x\right). \right. \tag{5}$$

Therefore,

$$A_n = 4\left((2n+1)\pi - 3(-1)^n\right)\frac{l^3}{\left(\left(\frac{2n+1}{2}\right)\pi\right)^4}. \tag{6}$$

Problem 2. As discussed in class about inner product spaces, we know that $\|f - \phi\|$ is minimal if we take the generalized Fourier coefficients. Meaning, if we call $c_n(x) = \cos(nx)$ and $s_n(x) = \sin(nx)$:

$$\begin{cases} a_0 = \dfrac{\langle \phi, c_0 \rangle}{\|c_0\|^2} = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} \phi(x) dx = \dfrac{\pi}{2}, \\ a_1 = \dfrac{\langle \phi, c_1 \rangle}{\|c_1\|^2} = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} \phi(x) \cos(x) dx = -\dfrac{4}{\pi}, \\ a_2 = \dfrac{\langle \phi, c_2 \rangle}{\|c_2\|^2} = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} \phi(x) \cos(2x) dx = 0, \\ b_1 = \dfrac{\langle \phi, s_1 \rangle}{\|s_1\|^2} = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} \phi(x) \sin(x) dx = 0, \\ b_2 = \dfrac{\langle \phi, s_2 \rangle}{\|s_2\|^2} = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} \phi(x) \sin(2x) dx = 0. \end{cases} \tag{7}$$

And hence $f(x) = \dfrac{\pi}{2} - \dfrac{4}{\pi} \cos(x)$.

Remark. We knew a priori that the b_n coefficients were zeros since ϕ is even.

Problem 3.

(i) Assume A is symmetric positive. Let λ be an eigenvalue of A with associated (non-zero) eigenfunction v. Then we have that λ is real since

$$\lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle Av, v \rangle = \langle v, Av \rangle = \langle v, \lambda v \rangle = \bar{\lambda} \langle v, v \rangle, \tag{8}$$

which implies that $\bar{\lambda} = \lambda$ since $\langle v, v \rangle \neq 0$.
Now we have

$$\lambda \langle v, v \rangle = \langle \lambda v, v \rangle = \langle Av, v \rangle \geq 0, \tag{9}$$

which gives that $\lambda \geq 0$ since $\langle v, v \rangle > 0$.

(ii) Let $f, g \in D$. Let's show that A is symmetric. By integration by part, and using the fact that $f(0) = f'(l) = 0$ and $g(0) = g'(l) = 0$, we get that

$$\begin{aligned}
\langle Af, g\rangle &= \int_0^l \left(-f''(x)\right)\overline{g(x)}dx \\
&= \left(-f'(x)\right)\overline{g(x)}\Big|_{x=0}^{x=l} - \int_0^l \left(-f'(x)\right)\overline{g'(x)}dx \\
&= \int_0^l f'(x)\overline{g'(x)}dx \\
&= f(x)\overline{g'(x)}\Big|_{x=0}^{x=l} - \int_0^l f(x)\overline{g''(x)}dx \\
&= \int_0^l f(x)\left(-\overline{g''(x)}\right)dx \\
&= \langle f, Ag\rangle.
\end{aligned} \tag{10}$$

Now let's prove that A is positive. We get that easily from the previous third equality, that is,

$$\langle Af, f\rangle = \int_0^l f'(x)\overline{f'(x)}dx = \int_0^l \left|f'(x)\right|^2 dx \geq 0. \tag{11}$$

(iii) We proceed in the same fashion as before. Let $f, g \in D$. Let's show that A is symmetric. By integration by part, and using the fact that $f(0) = f'(0) = f(0) = f'(l) = 0$ and $g(0) = g'(0) = g(l) = g'(l) = 0$, we get that

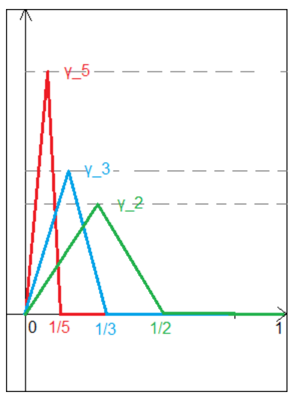
$$\begin{aligned}
\langle Af, g\rangle &= \int_0^l f^{(4)}(x)\overline{g(x)}dx \\
&= f'''(x)\overline{g(x)}\Big|_{x=0}^{x=l} - \int_0^l f'''(x)\overline{g'(x)}dx \\
&= -\int_0^l f'''(x)\overline{g'(x)}dx \\
&= -f''(x)\overline{g'(x)}\Big|_{x=0}^{x=l} + \int_0^l f''(x)\overline{g''(x)}dx \\
&= \int_0^l f''(x)\overline{g''(x)}dx \\
&= f'(x)\overline{g''(x)}\Big|_{x=0}^{x=l} - \int_0^l f'(x)\overline{g'''(x)}dx \\
&= -\int_0^l f'(x)\overline{g'''(x)}dx \\
&= -f(x)\overline{g'''(x)}\Big|_{x=0}^{x=l} + \int_0^l f(x)\overline{g^{(4)}(x)}dx \\
&= \int_0^l f(x)\overline{g^{(4)}(x)}dx \\
&= \langle f, Ag\rangle.
\end{aligned} \tag{12}$$

Now let's prove that A is positive. We get that easily from the fifth equality above, that is,

$$\langle Af, f\rangle = \int_0^l f''(x)\overline{f''(x)}dx = \int_0^l |f''(x)|^2 \, dx \geq 0. \tag{13}$$

Problem 4.

(i) Here are sketched some of the f_n's (say f_2, f_3 and f_5):



(ii) We need to show that for a fixed $x_0 \in [0,1]$, we have $f_n(x_0) \xrightarrow[n\to+\infty]{} 0$. If $x_0 = 0$, then $f_n(x_0) = 0$ for each n. If $0 < x_0 \leq 1$, there exists $n_0 \in \mathbb{N}$ such that $\dfrac{1}{n_0} < x_0$, and for all $n \geq n_0$, we get $f_n(x_0) = 0$ by definition of f_n. Therefore we have the result.

(iii) The convergence to zero is uniform if $\sup\limits_{x\in[0,1]} |f_n(x)| \xrightarrow[n\to+\infty]{} 0$. However, we see that

$$\sup_{x\in[0,1]} |f_n(x)| = \gamma_n \xrightarrow[n\to+\infty]{} +\infty, \tag{14}$$

which indeed does not converge to 0 as $n \longrightarrow +\infty$.

(iv) For $\gamma_n = n^{1/4}$, we have

$$\|f_n\|_{L^2}^2 = \int_0^1 f_n(x)^2 dx = 2(2n\gamma_n)^2 \frac{(2n)^{-3}}{3} = \frac{\gamma_n^2}{3n} = \frac{1}{3\sqrt{n}} \xrightarrow[n\to+\infty]{} 0. \tag{15}$$

Therefore we have the L^2 convergence.

(v) For $\gamma_n = n$, we have

$$\|f_n\|_{L^2}^2 = \frac{\gamma_n^2}{3n} = \frac{n}{3} \xrightarrow[n\to+\infty]{} +\infty. \tag{16}$$

Therefore f_n does not converge in L^2.

Problem 5.

(i) Using the fact that $\Delta = \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2}$ and the chain rule, we get that

$$\Delta(\phi u) = \phi\Delta u + u\Delta\phi + 2\sum_{j=1}^{3} \frac{\partial \phi}{\partial x_j}\frac{\partial u}{\partial x_j}. \tag{17}$$

Since ϕ is identically 1 near x_0, $\frac{\partial \phi}{\partial x_j}$ and $\Delta\phi$ are identically 0 near x_0. Therefore by defining

$$v = u\Delta\phi + 2\sum_{j=1}^{3} \frac{\partial \phi}{\partial x_j}\frac{\partial u}{\partial x_j}, \tag{18}$$

v vanishes near x_0 and has compact support since $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$.
Now from Problem Set 6, we have that $\Delta(E * w) = w$, where E is the fundamental solution for the Laplacian. Also we have $\Delta(E * w) = E * \Delta w$, and hence

$$w(x) = -\int_{\mathbb{R}^3} \frac{1}{4\pi|x-y|}\Delta_y(\phi(y)u(y))dy. \tag{19}$$

(ii) We want to show that $\int_{\mathbb{R}^3} \frac{1}{4\pi|y|}\phi(x-y)(\Delta u)(x-y)dy$ is $\mathcal{C}^\infty$. Since $\phi\Delta u$ is $\mathcal{C}^\infty$, it suffices to show that the integral is indeed integrable. The only problem that we may have is when $y = 0$. But since $\phi\Delta u$ is $\mathcal{C}^\infty$ and compactly supported, for fixed x, there R such that

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{1}{4\pi|y|}\phi(x-y)(\Delta u)(x-y)dy &= \lim_{\epsilon\to 0}\int_{\epsilon\le|y|\le R} \frac{1}{4\pi|y|}\phi(x-y)(\Delta u)(x-y)dy \\
&\le \lim_{\epsilon\to 0} C\int_{\epsilon\le|y|\le R} \frac{1}{4\pi|y|}dy \\
&= \lim_{\epsilon\to 0} C\int_{\epsilon}^{R}\int_0^{\pi}\int_0^{2\pi} \frac{1}{4\pi r}r^2\sin(\theta)d\phi d\theta dr \\
&= \lim_{\epsilon\to 0} C\int_{\epsilon}^{R} rdr \le C\frac{R^2}{2} < +\infty
\end{aligned} \tag{20}$$

Therefore the quantity is integrable and $\int_{\mathbb{R}^3} \frac{1}{4\pi|y|}\phi(x-y)(\Delta u)(x-y)dy$ is $\mathcal{C}^\infty$.

(iii) For $u \in \mathcal{D}'(\mathbb{R})$, we have, for any $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$,

$$\begin{aligned}
(\phi\Delta u)(\psi) &= \Delta u(\phi\psi) \\
&= u(\phi\Delta\psi) + u((\Delta\phi)\psi) + u\left(2\sum_{j=1}^{3}\frac{\partial\phi}{\partial x_j}\frac{\partial\psi}{\partial x_j}\right) \\
&= (\Delta(\phi u))(\psi) + (\Delta\phi)u(\psi) - 2\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial x_j}u\right)(\psi).
\end{aligned} \tag{21}$$

Let $v = -(\Delta\phi)u(\psi) + 2\sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\frac{\partial\phi}{\partial x_j}u\right)$. Then we see that $\Delta(\phi u) = \phi\Delta u + v$, and v is a compactly supported distribution that vanishes near x_0.

Now $\phi\Delta u = \phi f$, which is $\mathcal{C}^\infty$. By (ii), $-\frac{1}{4\pi|x|} * \phi\Delta u$ is then $\mathcal{C}^\infty$.

Let $g_x(y) = -\frac{1}{4\pi|x-y|}$, so that $v * -\frac{1}{4\pi|x|} = v(g_x)$. Since v is a compactly supported distribution that vanishes near x_0, g_x can be treated as $\mathcal{C}^\infty$ where x is near x_0, $v(g_x)$ makes sense and is $\mathcal{C}^\infty$.