This is a closed book, closed notes, no calculators exam.

There are 5 problems. Solve all of them. Write your solutions to problems 1 and 2 in blue book #1, and your solutions to problems 3, 4 and 5 in blue book #2. Within each book, you may solve the problems in any order. Total score: 100 points.

Problem 1. (20 points) Solve the PDE
\[ e^y u_x + u_y = u^2, \quad u(x,0) = x, \]
for small \(|y|\) (i.e. in a neighborhood of the \(x\) axis).

Problem 2.
(i) (12 points) On \( \mathbb{R}^n \times [0, \infty)_t \), solve the PDE
\[ -\Delta^2 u = u_t, \quad u(x,0) = \phi(x), \]
where \( \Delta^2 = \Delta_x(\Delta_x u) \), when \( \phi \) is a given Schwartz function. You may leave your answer as the (partial) inverse Fourier transform of a function (depending on \( \phi \)).
(ii) (8 points) Solve the equation when \( n = 1, \phi(x) = 1 - x^2 \) for \(-1 < x < 1\), and \( \phi(x) = 0 \) otherwise. You may leave your answer as the (partial) inverse Fourier transform of a function you have evaluated explicitly.

Problem 3.
(i) (10 points) Suppose that \( u \in D'(\mathbb{R}^2) \). State the definition of \( \partial u / \partial x \), and show that this is consistent with the standard definition of partial derivatives if \( u \) is given by some \( f \in C^1(\mathbb{R}^2) \) (i.e. if \( u = \iota f \)).
(ii) (10 points) Suppose that \( u \) is given by a piecewise continuous function \( f \) on \( \mathbb{R}^2 \), i.e.
\[ u(\phi) = \int_{\mathbb{R}^2} f(y)\phi(x,y) \, dx \, dy \]
for \( \phi \in C^\infty_c(\mathbb{R}^2) \). Show that \( \frac{\partial u}{\partial x} = 0 \).

Problem 4.
(i) (10 points) Find the general \( C^2 \) solution of the PDE
\[ u_{xx} - 4u_{xt} + 3u_{tt} = 0. \]
(ii) (10 points) Solve the initial value problem with initial condition
\[ u(x,2x) = \phi(x), \quad u_t(x,2x) = \psi(x), \]
with \( \phi, \psi \) given.

Problem 5. Consider the following equation on \( \mathbb{R} \times [0, \infty)_y \):
\[ Lu = -au_{xx} + 2bu_{xy} + u_{yy} = 0, \]
where \( a, b \) are constant, and suppose that \( L \) is hyperbolic.
(i) (4 points) State what hyperbolicity means in terms of \( a \) and \( b \).
(ii) (9 points) Suppose that \( u(x,0) \) and \( u_y(x,0) \) vanish for \( |x| \geq R \) and \( u \) is \( C^2 \), and \( a > 0 \). Let
\[ E(y) = \frac{1}{2} \int_\mathbb{R} (u_y(x,y)^2 + au_x(x,y)^2) \, dx. \]
Show that \( E \) is independent of \( y \). You may use without proof that this PDE has finite propagation speed. (This would be proved as in your homework problem.)
(iii) (7 points) Show that (among \( C^2 \) functions) the solution of the PDE \( Lu = 0 \) with \( u(x,0) = \phi(x), \ u_y(x,0) = \psi \), where \( \phi, \psi \) are given functions which vanish for \( |x| > R \), is unique.