MATH 220: MIDTERM
OCTOBER 29, 2009

This is a closed book, closed notes, no calculators exam.

There are 5 problems. Solve all of them. Write your solutions to problems 1 and 2 in blue book #1, and your solutions to problems 3, 4 and 5 in blue book #2. Within each book, you may solve the problems in any order. Total score: 100 points.

Problem 1. *(20 points)* Solve the PDE
$$e^y u_x + u_y = u^2,\ u(x,0) = x,$$
for small $|y|$ (i.e. in a neighborhood of the x axis).

Problem 2.

(i) *(12 points)* On $\mathbb{R}^n_x \times [0,\infty)_t$, solve the PDE
$$-\Delta_x^2 u = u_t,\ u(x,0) = \phi(x),$$
(where $\Delta_x^2 u = \Delta_x(\Delta_x u)$), when ϕ is a given Schwartz function. You may leave your answer as the (partial) inverse Fourier transform of a function (depending on ϕ).

(ii) *(8 points)* Solve the equation when $n = 1$, $\phi(x) = 1 - x^2$ for $-1 < x < 1$, and $\phi(x) = 0$ otherwise. You may leave your answer as the (partial) inverse Fourier transform of a function you have evaluated explicitly.

Problem 3.

(i) *(10 points)* Suppose that $u \in \mathcal{D}'(\mathbb{R}^2_{x,y})$. State the definition of $\frac{\partial u}{\partial x}$, and show that this is consistent with the standard definition of partial derivatives if u is given by some $f \in C^1(\mathbb{R}^2)$ (i.e. if $u = \iota_f$).

(ii) *(10 points)* Suppose that u is given by a piecewise continuous function f on $\mathbb{R}_y$, i.e. $u(\phi) = \int_{\mathbb{R}^2} f(y)\phi(x,y)\,dx\,dy$ for $\phi \in C_c^\infty(\mathbb{R}^2)$. Show that $\frac{\partial u}{\partial x} = 0$.

Problem 4.

(i) *(10 points)* Find the general C^2 solution of the PDE
$$u_{xx} - 4u_{xt} + 3u_{tt} = 0.$$

(ii) *(10 points)* Solve the initial value problem with initial condition
$$u(x,2x) = \phi(x),\ u_t(x,2x) = \psi(x),$$
with ϕ, ψ given.

Problem 5. Consider the following equation on $\mathbb{R}_x \times [0,\infty)_y$:
$$Lu \equiv -au_{xx} + 2bu_{xy} + u_{yy} = 0,$$
where a, b are constant, and suppose that L is hyperbolic.

(i) *(4 points)* State what hyperbolicity means in terms of a and b.

(ii) *(9 points)* Suppose that $u(x,0)$ and $u_y(x,0)$ vanish for $|x| \geq R$ and u is C^2, and $a > 0$. Let
$$E(y) = \frac{1}{2}\int_{\mathbb{R}} (u_y(x,y)^2 + au_x(x,y)^2)\,dx.$$
Show that E is independent of y. You may use without proof that this PDE has finite propagation speed. (This would be proved as in your homework problem.)

(iii) *(7 points)* Show that (among C^2 functions) the solution of the PDE $Lu = 0$ with $u(x,0) = \phi(x)$, $u_y(x,0) = \psi$, where ϕ, ψ are given functions which vanish for $|x| > R$, is unique.