MATH 220: MIDTERM – SOLUTIONS
OCTOBER 29, 2009

This is a closed book, closed notes, no calculators exam.

There are 5 problems. Solve all of them. Write your solutions to problems 1 and 2 in blue book #1, and your solutions to problems 3, 4 and 5 in blue book #2. Within each book, you may solve the problems in any order. Total score: 100 points.

Problem 1. *(20 points)* Solve the PDE

$$e^y u_x + u_y = u^2,\ u(x,0) = x,$$

for small $|y|$ (i.e. in a neighborhood of the x axis).

Solution. The characteristic ODEs are

$$\frac{dx}{ds} = e^y,\ x(r,0) = r,$$
$$\frac{dy}{ds} = 1,\ y(r,0) = 0,$$
$$\frac{dz}{ds} = z^2,\ z(r,0) = r.$$

From the ODE for y, $y(r,s) = s$, so the x ODE becomes $\frac{dx}{ds} = e^s$, hence $x(r,s) = e^s + r - 1$. Finally the z ODE gives (for $z \neq 0$)

$$-\frac{1}{z}|_0^s = s - 0,$$

so

$$r^{-1} - z(r,s)^{-1} = s,$$

and thus

$$z = \frac{1}{r^{-1} - s} = \frac{r}{1 - rs}.$$

Since $s = y$ and $r = x - e^s + 1 = x - e^y + 1$,

$$u(x,y) = \frac{x - e^y + 1}{1 - y(x - e^y + 1)}.$$

Problem 2.

(i) *(12 points)* On $\mathbb{R}^n_x \times [0,\infty)_t$, solve the PDE

$$-\Delta_x^2 u = u_t,\ u(x,0) = \phi(x),$$

(where $\Delta_x^2 u = \Delta_x(\Delta_x u)$), when ϕ is a given Schwartz function. You may leave your answer as the (partial) inverse Fourier transform of a function (depending on ϕ).

(ii) *(8 points)* Solve the equation when $n = 1$, $\phi(x) = 1 - x^2$ for $-1 < x < 1$, and $\phi(x) = 0$ otherwise. You may leave your answer as the (partial) inverse Fourier transform of a function you have evaluated explicitly.

Solution. Taking a partial Fourier transform in x, denoting the corresponding variable by ξ, gives

$$-|\xi|^4 \mathcal{F}_x u = \mathcal{F}_x(-\Delta_x^2 u) = \mathcal{F}_x(u_t) = \frac{d\mathcal{F}_x u}{dt},\ \mathcal{F}_x u(\xi,0) = (\mathcal{F}\phi)(\xi).$$

Solving the ODE yields

$$(\mathcal{F}_x u)(\xi,t) = e^{-|\xi|^4 t}(\mathcal{F}\phi)(\xi),$$

so

$$u = \mathcal{F}_\xi^{-1}\left(e^{-|\xi|^4 t}(\mathcal{F}\phi)(\xi)\right).$$

Evaluating $\mathcal{F}\phi$ explicitly in the special case for $\xi \neq 0$ (note that we are taking the Fourier transform of a continuous function of compact support, so the Fourier transform is continuous, i.e. its value at zero is just the limit as $\xi \to 0$, hence we do *not* need to consider $\xi = 0$ separately):

$$\begin{aligned}
\mathcal{F}\phi(\xi) &= \int_{-1}^{1}(1-x^2)e^{-ix\xi}\,dx \\
&= \frac{-1}{i\xi}e^{-ix\xi}(1-x^2)|_{-1}^{1} - \frac{2}{i\xi}\int_{-1}^{1} xe^{-ix\xi}\,dx \\
&= \frac{-2}{\xi^2}xe^{-ix\xi}|_{-1}^{1} + \frac{2}{\xi^2}\int_{-1}^{1} e^{-ix\xi}\,dx \\
&= \frac{-2}{\xi^2}(e^{i\xi}+e^{-i\xi}) - \frac{2}{i\xi^3}e^{-ix\xi}|_{-1}^{1} \\
&= \frac{-2}{\xi^2}(e^{i\xi}+e^{-i\xi}) + \frac{2}{i\xi^3}(e^{i\xi}-e^{-i\xi}) \\
&= \frac{-2}{\xi^3}\left((\xi+i)e^{i\xi}+(\xi-i)e^{-i\xi}\right).
\end{aligned}$$

Thus, the solution of the PDE is

$$u = \mathcal{F}_\xi^{-1}\left(\frac{-2}{\xi^3}e^{-|\xi|^4 t}\left((\xi+i)e^{i\xi}+(\xi-i)e^{-i\xi}\right)\right).$$

Problem 3.

(i) *(10 points)* Suppose that $u \in \mathcal{D}'(\mathbb{R}^2_{x,y})$. State the definition of $\frac{\partial u}{\partial x}$, and show that this is consistent with the standard definition of partial derivatives if u is given by some $f \in C^1(\mathbb{R}^2)$ (i.e. if $u = \iota_f$).

(ii) *(10 points)* Suppose that u is given by a piecewise continuous function f on $\mathbb{R}_y$, i.e. $u(\phi) = \int_{\mathbb{R}^2} f(y)\phi(x,y)\,dx\,dy$ for $\phi \in C_c^\infty(\mathbb{R}^2)$. Show that $\frac{\partial u}{\partial x} = 0$.

Solution. The definition of $\frac{\partial u}{\partial x}$ is

$$\partial_x u(\phi) = -u(\partial_x\phi),\ \ \phi \in C_c^\infty(\mathbb{R}^2).$$

If $u = \iota_f$ then

$$-u(\partial_x\phi) = -\iota_f(\partial_x\phi) = -\int f\partial_x\phi\,dx\,dy = \int \partial_x f\phi\,dx\,dy = \iota_{\partial_x f}(\phi),$$

so $\partial_x\iota_f = \iota_{\partial_x f}$ indeed.

In this case we have

$$\begin{aligned}
\partial_x u(\phi) = -u(\partial_x\phi) &= -\int_{\mathbb{R}^2} f(y)\partial_x\phi(x,y)\,dx\,dy \\
&= -\int_{\mathbb{R}} f(y)\left(\int_{\mathbb{R}}\partial_x\phi(x,y)\,dx\right)dy = 0
\end{aligned}$$

by the fundamental theorem of calculus, using that ϕ has compact support.

Problem 4.

(i) *(10 points)* Find the general C^2 solution of the PDE

$$u_{xx} - 4u_{xt} + 3u_{tt} = 0.$$

(ii) *(10 points)* Solve the initial value problem with initial condition

$$u(x,2x) = \phi(x),\ \ u_t(x,2x) = \psi(x),$$

with ϕ, ψ given.

Solution. We factor the operator as

$$\partial_x^2 - 4\partial_x\partial_t + 3\partial_t^2 = (\partial_x - \partial_t)(\partial_x - 3\partial_t).$$

The characteristics of $\partial_x - \partial_t$ are $x + t$ constant, while those of $\partial_x - 3\partial_t$ are $3x + t$ constant, so

$$u(x,t) = f(x+t) + g(3x+t)$$

certainly solves the PDE. That this is the general solution, would follow by a change of coordinates reducing it to the wave equation, in which case we have already shown the analogous formula. It can be also shown by solving the characteristic ODEs as in the practice exam solutions.

We proceed to solve the initial value problem by substituting in the initial conditions:

$$\phi(x) = u(x,2x) = f(3x) + g(5x),\ \ \psi(x) = u_t(x,2x) = f'(3x) + g'(5x).$$

The first equation gives

$$\phi'(x) = 3f'(3x) + 5g'(5x),$$

so

$$\begin{aligned} 2f'(3x) &= 5\psi(x) - \phi'(x), \\ 2g'(5x) &= \phi'(x) - 3\psi(x) \end{aligned}$$

and thus

$$\begin{aligned} f'(s) &= \frac{5}{2}\psi(s/3) - \frac{1}{2}\phi'(s/3) \\ g'(s) &= \frac{1}{2}\phi'(s/5) - \frac{3}{2}\psi(s/5). \end{aligned}$$

Integrating yields

$$\begin{aligned} f(s) &= -\frac{3}{2}\phi(s/3) + \frac{5}{2}\int_0^s \psi(\sigma/3)\,d\sigma + A = -\frac{3}{2}\phi(s/3) + \frac{15}{2}\int_0^{s/3} \psi(\sigma)\,d\sigma + A, \\ g(s) &= \frac{5}{2}\phi(s/5) - \frac{3}{2}\int_0^s \psi(\sigma/5)\,d\sigma + B = \frac{5}{2}\phi(s/5) - \frac{15}{2}\int_0^{s/5} \psi(\sigma)\,d\sigma + B. \end{aligned}$$

Substituting into $\phi(x) = f(3x) + g(5x)$ gives $B = -A$. Thus,

$$u(x,t) = -\frac{3}{2}\phi((x+t)/3) + \frac{5}{2}\phi((3x+t)/5) + \frac{15}{2}\int_{(3x+t)/5}^{(x+t)/3} \psi(\sigma)\,d\sigma.$$

Problem 5. Consider the following equation on $\mathbb{R}_x \times [0,\infty)_y$:

$$Lu \equiv -au_{xx} + 2bu_{xy} + u_{yy} = 0,$$

where a, b are constant, and suppose that L is hyperbolic.

(i) *(4 points)* State what hyperbolicity means in terms of a and b.

(ii) *(9 points)* Suppose that $u(x,0)$ and $u_y(x,0)$ vanish for $|x| \geq R$ and u is C^2, and $a > 0$. Let

$$E(y) = \frac{1}{2}\int_{\mathbb{R}} (u_y(x,y)^2 + au_x(x,y)^2)\,dx.$$

Show that E is independent of y. You may use without proof that this PDE has finite propagation speed. (This would be proved as in your homework problem.)

(iii) *(7 points)* Show that (among C^2 functions) the solution of the PDE $Lu = 0$ with $u(x,0) = \phi(x)$, $u_y(x,0) = \psi$, where ϕ, ψ are given functions which vanish for $|x| > R$, is unique.

Solution. The PDE being hyperbolic says that $\det \begin{bmatrix} -a & b \\ b & 1 \end{bmatrix} < 0$, i.e. $-a - b^2 < 0$, so $a > -b^2$.

By the finite speed of propagation, for each $T > 0$, there is an $R' > 0$ such that $u(x,t) = 0$ if $|y| < T$ and $|x| > R'$. Thus, the integral defining E makes sense and

$$\begin{aligned} \frac{dE}{dy}(y) &= \int_{\mathbb{R}} (u_y u_{yy} + a u_x u_{xy})\, dx \\ &= \int_{\mathbb{R}} (u_y u_{yy} - a u_{xx} u_y)\, dx = \int_{\mathbb{R}} u_y(-2b) u_{xy}\, dx \\ &= (-b) \int_{\mathbb{R}} (u_y^2)_x\, dx = 0, \end{aligned}$$

where the second equality followed by integration by parts, the third from the PDE, and the last from the fundamental theorem of calculus.

If one has two solutions, $u^{(j)}$, $j = 1, 2$, then the difference $u = u^{(1)} - u^{(2)}$ satisfies the homogeneous equation with vanishing initial data, so $E(0) = 0$, hence $E(y) = 0$ for all y. Since $a > 0$, u_x and u_y vanish identically, so u is a constant. Since u vanishes at $y = 0$, u is identically zero, and this $u^{(1)} = u^{(2)}$.

It is instructive to see the role of the assumption $a > 0$. During the integration by parts argument, one ends up with the integral of $(Lu)u_y$. The role of $a > 0$ is the the characteristics of the vector field ∂_y in this expression behave differently with respect to the characteristics of L. Namely, the characteristics of L come from the factorization

$$L = (\partial_y - \alpha_+ \partial_x)(\partial_y - \alpha_- \partial_x),$$

where

$$\alpha_\pm = -b \pm \sqrt{b^2 + a}.$$

If $a > 0$ then $\alpha_\pm$ have different signs, while if $a < 0$ they have the same sign, finally if $a = 0$, one of them vanishes. Correspondingly, the characteristics are

$$\alpha_+ y + x = \text{const}, \text{ resp. } \alpha_- y + x = \text{const},$$

have slopes with opposite signs if $a > 0$ and the same sign if $a < 0$. Considering, say characteristics through the origin, the characteristics of L divide $\mathbb{R}^2$ into four quadrants. Now, the characteristics of ∂_y, given by $x = \text{const}$ lie either in the same quadrants as the initial data axis ($y = 0$), if $a < 0$, or in different quadrants, if $a > 0$. This explains why one would expect the energy E to be positive definite if $a > 0$, but not otherwise.