CME 303/MATH 220: MIDTERM
NOVEMBER 1, 2018

This is a closed book, closed notes, no electronic devices exam.

There are 5 problems. **Solve Problems 1-3 and one of Problems 4 and 5.** Write your solutions to problems 1 and 2 in blue book #1, and your solutions to problems 3, 4 and 5 in blue book #2. Within each book, you may solve the problems in any order. Total score: 100 points.

Problem 1. *(25 points)*

(i) Solve the PDE

$$(x-1)u_x + u_y = 2u,\ u(0,y) = e^y,$$

on as large of a subset of $\mathbb{R}^2$ containing the y-axis as you can. Do all characteristics intersect the y-axis, and do so in a unique point, non-tangentially? What does this mean for the solutions of the initial value problem of the PDE? Sketch the characteristics.

(ii) Regarding the region of solvability and the characteristics mentioned in part (i), what changes if the right hand side of the PDE is replaced by u^2:

$$(x-1)u_x + u_y = u^2,\ u(0,y) = e^y\ ?$$

You do not need to give a full solution of the PDE, but you need to justify your answer.

Solution. (i) The PDE is linear. The associated vector field in $\mathbb{R}^2$ is

$$V(x,y) = (x-1, 1),$$

and the surface on which we prescribe the function values is

$$\Gamma = \{(0,r) : r \in \mathbb{R}\}.$$

It can be readily seen that the vector field is non-characteristic, i.e., at no point $(x,y) \in \Gamma$ the vector $V(x,y)$ is tangent to Γ at (x,y).

We write $\Gamma(r) = (\Gamma_x(r), \Gamma_y(r)) = (0,r)$. The equations for the projected characteristics are

$$\begin{aligned} \dot{x}_r(s) &= x_r(s) - 1, & x_r(0) &= 0, \\ \dot{y}_r(s) &= 1, & y_r(0) &= r, \end{aligned}$$

and

$$\dot{v}_r(s) = 2v_r(s), \quad v_r(0) = e^r,$$

where $v_r(s) = u(x_r(s), y_r(s))$. Writing the first equation as $\dot{(x-1)}_r(s) = x_r(s) - 1$, $x_r(0) - 1 = -1$, It follows that

$$\begin{aligned} x_r(s) - 1 &= -e^s, \\ y_r(s) &= s + r, \\ v_r(s) &= e^r e^{2s}, \end{aligned}$$

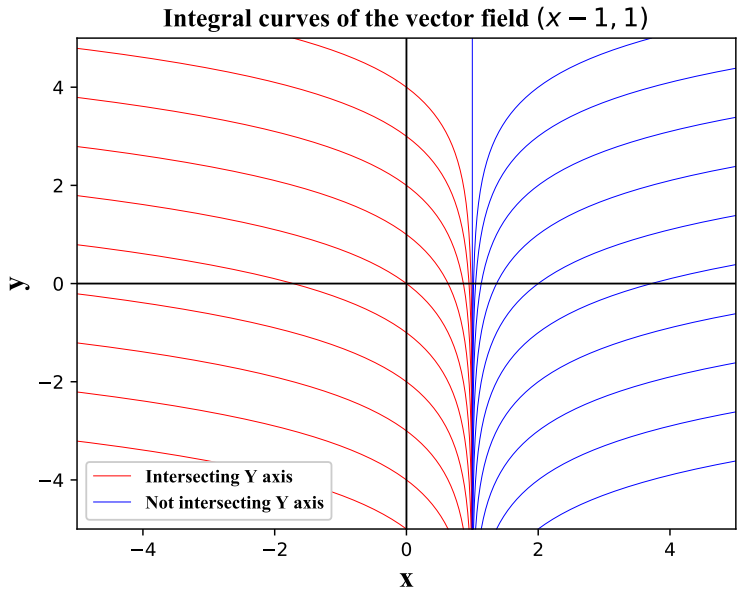

FIGURE 1. Characteristics for the PDE in Problem 1.

so $x_r(s) = 1 - e^s$.

To check whether we can perform the change of variables $(r, s) \to (x, y)$ note that

$$\det \begin{bmatrix} \partial_r x_r(s) & \partial_r y_r(s) \\ \partial_s x_r(s) & \partial_s y_r(s) \end{bmatrix} = \det \begin{bmatrix} 0 & 1 \\ -e^s & 1 \end{bmatrix} = e^s \neq 0,$$

for any (r, s), which implies the inverse map exists locally, so in particular locally near the y axis we can invert it. This gives, for $x < 1$ (which includes the y axis)

$$\begin{aligned} s &= \log(1 - x), \\ r &= y - s = y - \log(1 - x) \end{aligned}$$

so

$$u(x, y) = e^{y - \log(1-x)} e^{2\log(1-x)} = e^{y + \log(1-x)} = e^y(1 - x), \ x < 1.$$

Note that the x component of the vector field giving the characteristics is -1 along the y axis, so the integral curves are not tangential to the y axis (when they cross it). Moreover, our solution shows that the crossing happens, for the integral curves in $x < 1$, in a unique location.

Note that at $x = 1$, the vector field giving the characteristics is non-vanishing, but has vanishing x component, so $x = 1$ is a characteristic curve. Since characteristics cannot cross, characteristics from $x > 1$ do not reach the y axis, and thus the solution will not be determined along them.

(ii) With the right hand side changed to u^2, the PDE is still semilinear, and the same method of solution works. In particular, the characteristics, and thus their behavior, are unchanged. However, the region of solvability can,

and actually does, change, because the v ODE becomes non-linear, and the solutions may blow up for finite s.

□

Problem 2.

(i) *(7 points)* Find the general C^2 solution of the PDE

$$u_{xx} - 5u_{xt} + 4u_{tt} = 0.$$

(ii) *(7 points)* Solve the initial value problem with initial condition

$$u(x,0) = \phi(x),\ u_t(x,0) = \psi(x),$$

with ϕ, ψ given C^2 functions.

(iii) *(6 points)* If $\phi(x)$ and $\psi(x)$ are 0 for $|x| \geq 1$, where can you say for sure that u is 0? Explain the geometric meaning.

(iv) *(5 points)* Give an example of a discontinuous distributional solution of the PDE, and explain why this solves the PDE (you may use results from the problem sets).

Solution: (i) We first determine the type of second-order operator. Note that

$$\det \begin{bmatrix} 1 & -5/2 \\ -5/2 & 4 \end{bmatrix} = 4 - \frac{25}{4} = -\frac{9}{4} < 0,$$

and the PDE is hyperbolic. We can factor it as the product of two first-order operators as

$$(a\partial_x + b\partial_t)(c\partial_x + d\partial_t) = ac\partial_{xx} + (ad + bc)\partial_{xt} + bd\partial_{tt},$$

from where it follows that

$$\begin{aligned} ac &= 1, \\ ad + bc &= -5, \\ bd &= 4. \end{aligned}$$

Choosing $a = 1$ we see that $c = 1$, $d + b = -5$, and $bd = 4$. We conclude $b = -4$ and $d = -1$ (or vice versa) and thus

$$\partial_x^2 - 5\partial_x\partial_t + 4\partial_t^2 = (\partial_x - 4\partial_t)(\partial_x - \partial_t).$$

As we have seen throughout the class, the general form of the C^2 solution is

$$u(x,t) = f(4x + t) + g(x + t),$$

for some C^2 functions f and g.

(ii) Since the general solution to the PDE is

$$u(x,t) = f(4x + t) + g(x + t),$$

we only need to find $f, g \in C^2(\mathbb{R})$ such that u satisfies the desired conditions. Since

$$u_t(x,t) = f'(4x + t) + g'(x + t),$$

we obtain the system of equations

$$\begin{aligned} \phi(x) &= f(4x) + g(x), \\ \psi(x) &= f'(4x) + g'(x), \end{aligned}$$

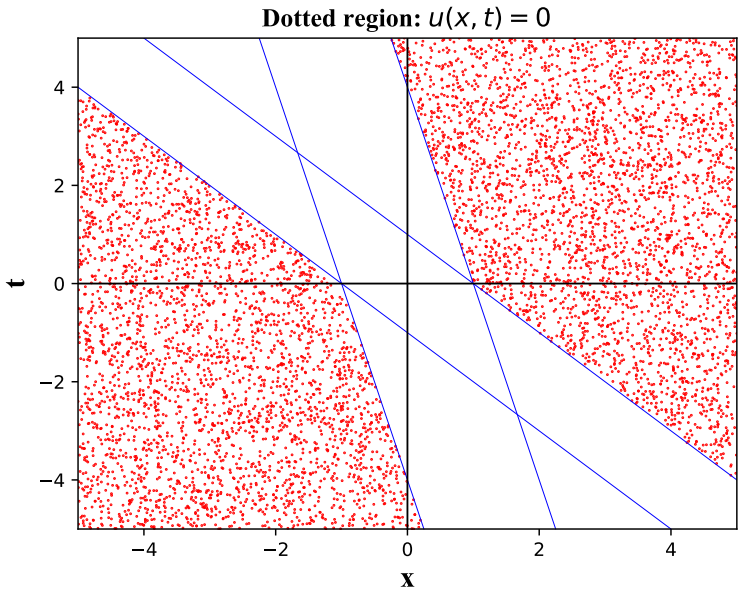

FIGURE 2. Characteristics for Problem 2; the solution is 0 in the dotted region.

and by differentiating the first equation we obtain the system

$$\phi'(x) = 4f'(4x) + g'(x),$$
$$\psi(x) = f'(4x) + g'(x).$$

Subtracting the second from the first leads to $\phi'(x) - \psi(x) = 3f'(4x)$ from where it follows that

$$f(x)-f(0) = \frac{1}{3}\int_0^x \phi'\left(\frac{1}{4}y\right)\,dy - \frac{1}{3}\int_0^x \psi\left(\frac{1}{4}y\right)\,dy = \frac{4}{3}\phi\left(\frac{1}{4}x\right) - \frac{4}{3}\phi(0) - \frac{4}{3}\int_0^{\frac{1}{4}x}\psi(y)\,dy.$$

Subtracting the first from 4 times the second leads to $-\phi'(x) + 4\psi(x) = 3g'(x)$ and

$$g(x) - g(0) = -\frac{1}{3}\int_0^x \phi'(y)\,dy + \frac{4}{3}\int_0^x \psi(y)\,dy = -\frac{1}{3}\phi(x) + \frac{1}{3}\phi(0) + \frac{4}{3}\int_0^x \psi(y)\,dy.$$

Consequently

$$u(x,t) = f(0)+g(0)-\phi(0)+\frac{4}{3}\phi\left(x+\frac{1}{4}t\right) - \frac{1}{3}\phi(x+t) - \frac{4}{3}\int_0^{x+t/4}\psi(y)\,dy + \frac{4}{3}\int_0^{x+t}\psi(y)\,dy.$$

Note that $g(0) + f(0) = \phi(0)$ and thus

$$u(x,t) = \frac{4}{3}\phi\left(x+\frac{1}{4}t\right) - \frac{1}{3}\phi(x+t) + \frac{4}{3}\int_{x+t/4}^{x+t}\psi(y)\,dy.$$

(iii) Note that for $t > 0$, $x + t/4 < x + t$. Thus, if ϕ, ψ are 0 outside $(-1, 1)$, then we can say for sure that the solution vanishes at points (x, t) for which either $x + t/4 \geq 1$ or $x + t \leq -1$. (For $t < 0$ the order reverses, so $x + t \geq 1$ or $x + t/4 \leq -1$ gives the region.) Geometrically, this means the set of

points outside the characteristics that cross the $(-1, 1)$ interval on the x axis.

(iv) From Problem Set 2, Problem 3, we know that if f is piecewise continuous, $u(x,t) = f(x+t)$, then u solves $u_x - u_t = 0$ in the distributional sense. Thus,

$$(\partial_x^2 u - 5\partial_x\partial_t u + 4\partial_t^2 u) = (\partial_x - 4\partial_t)(\partial_x - \partial_t)u = (\partial_x - 4\partial_t)0 = 0,$$

so u solves the wave equation. For instance, we can take f to be the step function, $f(s) = 1$ for $s > 0$, $f(s) = 0$ for $s < 0$.

□

Problem 3. *(25 points)* Consider the (real-valued) heat equation on $[0, \ell]_x \times [0, \infty)_t$ with Dirichlet boundary conditions:

$$u_t = (k(x)u_x)_x, \ u(0,t) = 0 = u(\ell, t),$$

where $k > 0$ is allowed to depend on x, $k \in C^1([0, \ell])$. Assume throughout that u is C^2. Let

$$E(t) = \frac{1}{2}\int_0^\ell (u(x,t)^2 + k(x)u_x(x,t)^2)\, dx.$$

(i) Show that E is a decreasing function of t: $E(t_1) \geq E(t_2)$ if $t_1 \leq t_2$.
(ii) Show the solution of the variable coefficient heat equation (under the conditions mentioned above) with given initial condition, $u(x,0) = \phi(x)$, is unique.
(iii) Give a stability statement for the solution of the variable coefficient heat equation.
(iv) Are your conclusions also valid for Neumann boundary conditions $u_x(0,t) = 0 = u_x(\ell, t)$ in place of the Dirichlet boundary conditions?

Solution: (i) First note E is well-defined for $t > 0$ as the integral is over a bounded domain, and the integrand is C^2. Since the boundary condition implies $u_t(0,t) = u_t(\ell, t) = 0$ for all $t > 0$, exchanging the derivative with

respect to t and the integral shows that

$$\begin{aligned}
E'(t) &= \frac{1}{2}\frac{d}{dt}\int_0^\ell (u(x,t)^2 + k(x)u_x(x,t)^2)\,dx \\
&= \frac{1}{2}\int_0^\ell \frac{d}{dt}(u_t(x,t)^2 + k(x)u_x(x,t)^2)\,dx \\
&= \int_0^\ell (u(x,t)u_t(x,t) + k(x)u_x(x,t)u_{xt}(x,t))\,dx \\
&= \int_0^\ell u(x,t)(k(x)u_x(x,t))_x\,dx + \int_0^\ell \underbrace{(u_t(x,t) - (k(x)u_x(x,t))_x)}_{=0 \text{ as } u \text{ solves the PDE}} u(x,t)\,dx \\
&\quad + \int_0^\ell (k(x)u_x(x,t)u_t(x,t))_x\,dx - \int_0^\ell (k(x)u_x(x,t))_x u_t(x,t)\,dt \\
&= \int_0^\ell (u(x,t)k(x)u_x(x,t))_x\,dx - \int_0^\ell k(x)u_x(x,t)^2\,dx \\
&\quad + \int_0^\ell (k(x)u_x(x,t)u_t(x,t))_x\,dx - \int_0^\ell (k(x)u_x(x,t))_x u_t(x,t)\,dt \\
&= \underbrace{u(x,t)k(x)u_x(x,t)|_{x=0}^\ell}_{=0 \text{ by BC}} - \int_0^\ell k(x)u_x(x,t)^2\,dx \\
&\quad + \underbrace{k(x)u_x(x,t)u_t(x,t)|_{x=0}^\ell}_{=0 \text{ by BC, note } u=0 \text{ at } x=0,\ell \Rightarrow u_t = 0 \text{ there}} - \int_0^\ell u_t(x,t)^2\,dt + \int_0^\ell \underbrace{(u_t - (k(x)u_x(x,t))_x)}_{=0 \text{ as } u \text{ solves the PDE}} u_t(x,t)\,dt \\
&= -\int_0^\ell k(x)u_x(x,t)^2\,dx - \int_0^\ell u_t(x,t)^2\,dt \le 0,
\end{aligned}$$

and thus E is a decreasing function of t.

(ii) If we fix the initial conditions, $u(x,0) = \phi(x)$ for $x \in (0,\ell)$, then if u_1 and u_2 are two solutions to the PDE, then $w = u_1 - u_2$ is a solution to the homogeneous problem

$$w_t = (k(x)w_x)_x, \quad w(0,t) = w(\ell,t) = 0, \quad w(x,0) = 0.$$

In this case

$$E(0) = \frac{1}{2}\int_0^\ell (w(x,0)^2 + k(x)^2 w_x(x,0)^2)\,dx = 0,$$

and by (i) we conclude $E(t) \le 0$. Since $0 \le E(t)$ it follows that $E(t) \equiv 0$ for $t \ge 0$ and we have

$$\begin{aligned}
0 &= E(t) \\
&= \frac{1}{2}\int_0^\ell (w(x,t)^2 + k(x)^2 w_x(x,t)^2)\,dx \\
&\ge \frac{1}{2}\int_0^\ell w(x,t)^2\,dx
\end{aligned}$$

which implies $w(x,t) = 0$ for $(x,t) \in [0,\ell]_x \times [0,\infty)_t$. We conclude $u_1 = u_2$ and the solution to the PDE with given initial conditions is unique.

(iii) The stability statement is that if ϕ_1 and ϕ_2 are close, then the corresponding solutions u_1 and u_2 are also close. This is true in the sense of E: as above, take $w = u_1 - u_2$, and conclude that

$$\frac{1}{2}\int_0^\ell ((u_1(x,t) - u_2(x,t))^2 + k(x)((u_1)_x(x,t) - (u_2)_x(x,t))^2)\,dx = E(t)$$
$$\leq E(0) = \frac{1}{2}\int_0^\ell ((\phi_1(x) - \phi_2(x))^2 + k(x)(\phi_1'(x) - \phi_2'(x))^2)\,dx.$$

(iv) The boundary conditions also give the vanishing of the boundary terms for Neumann boundary conditions, thus all conclusions remain valid.

□

Problem 4. Consider the equation

$$u_t + uu_x = 0,\ u(x,0) = \phi(x),\ t \geq 0.$$

(i) *(6 points)* State the definition of u being a weak solution of this PDE.
(ii) *(10 points)* Suppose $c \in \mathbb{R}$, and $u(x,t) = 1$ if $x < ct$ and $u(x,t) = 0$ if $x > ct$. For what value of c does this give a weak solution of the PDE (for suitable initial condition)? Derive this directly from the definition of a weak solution, i.e. derive the Rankine-Hugoniot condition in this case.
(iii) *(9 points)* Suppose the initial condition is

$$\phi(x) = \begin{cases} 1, & x < 0, \\ 1 - x, & 0 < x < 1, \\ 0, & x > 1. \end{cases}$$

Will the solution develop a shock in $t > 0$? Why/why not?

Solution: (i) First note the PDE corresponds to a conservation law with $f(u) = u^2/2$. We say u is a weak solution of the PDE if

$$\iint_{\mathbb{R}_x \times [0,\infty)_t} (u\psi_t + f(u)\psi_x)\,dxdt + \int_{\mathbb{R}} \phi(x)\psi(x,0)\,dx = 0,$$

for all $\psi \in C^\infty(\mathbb{R}_x \times [0,\infty)_t)$ (which is the same as for all $\psi \in C^\infty(\mathbb{R}^2)$).

(ii) We proceed in the same way as in Problem 4, Homework 2. Define $\xi(t) = ct$ and

$$\Omega_- = \{(x,t) : x < \xi(t)\},$$
$$\Omega_+ = \{(x,t) : x > \xi(t)\},$$
$$\Gamma = \{(x,t) : x = \xi(t)\}.$$

By construction the restriction to u to $\Omega_\pm$ is constant. In particular

$$f(u) = \begin{cases} \frac{1}{2} & \text{if } (x,t) \in \Omega_-, \\ 0 & \text{if } (x,t) \in \Omega_+. \end{cases}$$

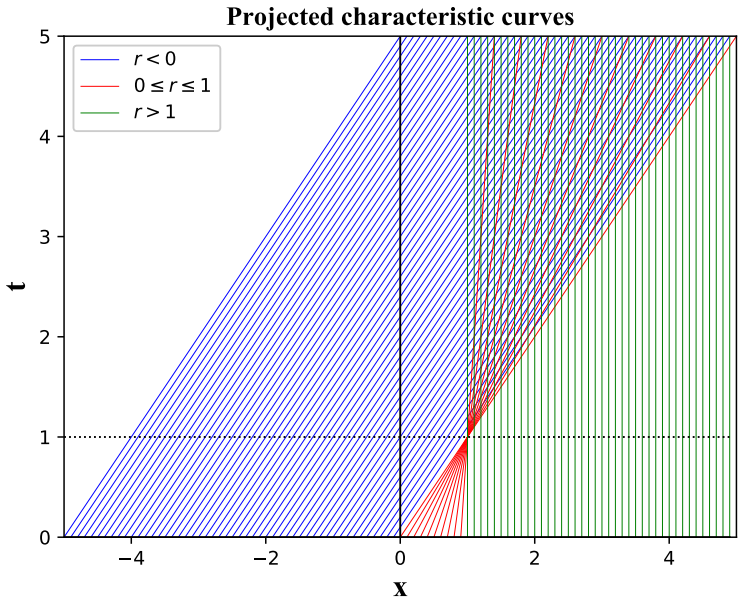

FIGURE 3. Characteristics for the PDE in Problem 4.

Let $\phi \in C^\infty(\mathbb{R}_x \times (0,\infty)_t)$. If u is a weak solution then

$$\begin{aligned} 0 &= \iint (u\phi_t + f(u)\phi_x)\,dxdt \\ &= \int_{\Omega_-} \left(\phi_t + \frac{1}{2}\phi_x\right)\,dxdt \\ &= \int_\Gamma \phi\nu_t\,dS + \frac{1}{2}\int_\Gamma \phi\nu_x\,dS \\ &= \int_\Gamma \left(\nu_t + \frac{1}{2}\nu_x\right)\phi\,dS, \end{aligned}$$

where $\nu = (\nu_x, \nu_t)$ is the outward-pointing unit normal vector *to* Ω_-. We can parameterize Γ using the function $\gamma(s) = (cs, s)$. From this it follows that $\dot\gamma(s) = (c, 1)$ and ν is a positive multiple of $(1, -c)$, thus $\frac{1}{\sqrt{1+c^2}}(1,-c)$. Therefore

$$\begin{aligned} 0 &= \int \left(-c + \frac{1}{2}\right)\phi(cs,s)\,ds \\ &= \left(-c + \frac{1}{2}\right)\int \phi(cs,s)\,ds, \end{aligned}$$

for any such ϕ. This is only possible if $c = 1/2$. Note that this is the Rankine-Hugoniot condition.

(iii) This is a quasilinear equation for which we impose initial conditions on the surface

$$\Gamma = \{(r,0) : r \in \mathbb{R}\},$$

and we parameterize it as $\Gamma(r) = (r, 0)$. We obtain the system of ODEs

$$\begin{aligned} \dot{t}_r(s) &= 1, & t_r(0) &= 0, \\ \dot{x}_r(s) &= z_r(s), & x_r(0) &= r, \\ \dot{z}_r(s) &= 0, & z_r(0) &= \phi(r), \end{aligned}$$

from where it readily follows that $t_r(s) = s$, $z_r(s) = \phi(r)$, and thus $x_r(s) = s\phi(r) + r$. Consequently $x_r(t) = t\phi(r) + r$. We know the solution will develop a shock if the characteristics intersect. Notice that for $0 < r < 1$, $\phi(r) = 1 - r$, so $x_r(t) = t(1 - r) + r = t + (1 - t)r$. Thus, when $t = 1$, all of the projected characteristics with initial point for $r \in (0, 1)$ go through $x = 1$, i.e. intersect, and thus shocks will necessarily develop.

□

Problem 5.

(i) *(5 points)* State the definition of a function f being Schwartz: $f \in \mathcal{S}(\mathbb{R}^n)$.

(ii) *(7 points)* Show that for f Schwartz, the Fourier transform of $f_a(x) = f(x - a)$ is $e^{-ia\cdot\xi}(\mathcal{F}f)(\xi)$, and use this to show that the Fourier transform of $D_j f = \frac{1}{i}\partial_j f$ is $\xi_j(\mathcal{F}f)(\xi)$.

(iii) *(7 points)* On $\mathbb{R}$, use the Fourier transform (by taking the F.T. of both sides) to solve the ODE

$$-\frac{d^2}{dx^2}u + u = e^{-x^2/2}.$$

You may leave your answer as the inverse Fourier transform of a function. You may use that the Fourier transform of $f(x) = e^{-ax^2}$, $a > 0$, is $(\pi/a)^{1/2}e^{-\xi^2/(4a)}$.

(iv) *(6 points)* Is the solution of the ODE obtained by the Fourier transform unique? How do you reconcile this with the fact that the ODE existence and uniqueness theorem guarantees at least local solvability and uniqueness to the initial value problem, where one also imposes $u(0) = c_0$, $u'(0) = c_1$, with $c_0, c_1 \in \mathbb{C}$ arbitrary? (*Hint:* what are the solutions of the homogeneous ODE?)

Solution: We break down the solutions by parts.

(i) A function $f \in C^\infty(\mathbb{R}^n)$ is in $\mathcal{S}(\mathbb{R}^n)$ if

$$\sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta f(x)| < \infty,$$

for any multiindices α, β.

(ii) We have

$$\begin{aligned} \mathcal{F}f_a(x) &= \int e^{-ix\cdot\xi} f(x - a)\, dx = e^{-ia\cdot\xi} \int e^{-i(x-a)\cdot\xi} f(x - a)\, dx \\ &= e^{-ia\cdot\xi} \int e^{-ix\cdot\xi} f(x)\, dx = e^{-ia\cdot\xi}(\mathcal{F}f)(\xi). \end{aligned}$$

Then taking $a = -he_j$, $e_j \in \mathbb{R}^n$ the jth unit vector,

$$(\mathcal{F}h^{-1}(f_a - f))(\xi) = h^{-1}(e^{-ia\cdot\xi} - 1)(\mathcal{F}f)(\xi) = h^{-1}(e^{ih\xi_j} - 1)(\mathcal{F}f)(\xi)$$

and the right hand side converges to $\frac{d}{dh}e^{ih\xi_j}|_{h=0}(\mathcal{F}f)(\xi) = i\xi_j(\mathcal{F}f)(\xi)$ as $h \to 0$. On the other hand,

$$h^{-1}(f_a - f) = h^{-1}(f(x + he_j) - f(x)),$$

so the argument of the Fourier transform on the left hand side converges to $\partial_j f$, from where we conclude $(\mathcal{F}D_j f)(\xi) = \xi_j(\mathcal{F}f)(\xi)$.

One should be somewhat careful (but such a discussion was not expected on the exam) in concluding that the Fourier transform of the limit, $\partial_j f$, is the limit of the Fourier transforms, but this follows easily by writing the argument of the Fourier transform as

$$h^{-1}(f(x + he_j) - f(x)) = \int_0^1 \partial_j f(x + the_j)\, dt$$

using the fundamental theorem of calculus (or the basic version of Taylor's theorem with integral remainder). Indeed, for $h > 0$, the integral over $\mathbb{R}^n$ and over $[0, 1]$ can be interchanged (Fubini), as can the integral over $[0, 1]$ and the limit ($[0, 1] \times \mathbb{R}^n \ni (t, x) \mapsto \partial_j f(x + the_j)$ is uniformly Schwartz in h, so $[0, 1] \ni t \mapsto \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\partial_j f(x + the_j)\, dx$ converges uniformly as $h \to 0$ to the constant function $[0, 1] \ni t \mapsto \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\partial_j f(x)\, dx$):

$$\lim_{h\to 0} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} h^{-1}(f(x + he_j) - f(x))\, dx = \lim_{h\to 0} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \int_0^1 \partial_j f(x + the_j)\, dt\, dx$$
$$= \lim_{h\to 0} \int_0^1 \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\partial_j f(x + the_j)\, dx\, dt = \int_0^1 \lim_{h\to 0} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\partial_j f(x + the_j)\, dx\, dt,$$

and then inside $\int_0^1$, the limit can be interchanged with the integral on $\mathbb{R}^n$ (as $\mathbb{R}^n \ni x \to \partial_j f(x + the_j)$ is uniformly bounded in $\mathcal{S}(\mathbb{R}^n)$ as $h \to 0$) in

$$\lim_{h\to 0} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\partial_j f(x + the_j)\, dx = \int_{\mathbb{R}^n} \lim_{h\to 0} e^{-ix\cdot\xi}\partial_j f(x + the_j)\, dx = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\partial_j f(x)\, dx$$

by standard limit theorems, giving the conclusion.

(iii) By taking Fourier transform on both sides we have

$$\xi^2(\mathcal{F}u)(\xi) + (\mathcal{F}u)(\xi) = (2\pi)^{1/2}e^{-\xi^2/2} \quad \Rightarrow \quad (\mathcal{F}u)(\xi) = \frac{(2\pi)^{1/2}}{1 + \xi^2}\, e^{-\xi^2/2},$$

which shows $\mathcal{F}u$ is in fact in $\mathcal{S}(\mathbb{R})$. Therefore we can take inverse Fourier transform to obtain

$$u(x) = \frac{1}{(2\pi)^{1/2}} \int \frac{e^{-\xi^2/2}}{1 + \xi^2} e^{ix\xi}\, d\xi,$$

and conclude that $u \in \mathcal{S}(\mathbb{R})$.

(iv) Using the Fourier transform we obtain a unique solution which is also Schwartz. Note this leaves no degrees of freedom to find a solution, not even *locally*, for a given initial condition. However, note that the solution of the ODE for given initial conditions is the sum of a solution to the homogeneous problem plus a particular solution to the inhomogeneous problem. Note the solutions to the homogeneous problem are

$$-\frac{d^2u}{dx^2} + u = 0 \quad \Rightarrow \quad u(x) = Ae^x + Be^{-x},$$

for some $A, B \in \mathbb{C}$. It can be seen that u is *not* Schwartz unless $A, B = 0$, as it does not decay as $x \to \infty$, resp. as $x \to -\infty$ when $A \neq 0$, resp. $B \neq 0$. Therefore taking Fourier transform allows us only to find a particular solution to the inhomogeneous problem when the inhomogeneous term is a Schwartz function: the exponentials are not in the space of functions accessed by the Fourier transform.

□