MATH 220: Problem Set 3
Solutions

Problem 1. Let \( \psi \in C(\mathbb{R}) \) be given by:

\[
\psi(x) = \begin{cases} 
0, & x < -1, \\
1 + x, & -1 < x < 0, \\
1 - x, & 0 < x < 1, \\
0, & x > 1,
\end{cases}
\]

so that it verifies \( \psi \geq 0, \) \( \psi(x) = 0 \) if \( |x| \geq 1 \) and \( \int_{\mathbb{R}} \psi(x)dx = 1. \)
Consider \( (\psi_j)_{j \geq 1} \) constructed as \( \psi_j(x) = j\psi(jx), \) so that \( \psi_j(x) = 0 \) if \( |x| \geq 1/j \), and \( \int_{\mathbb{R}} \psi_j(x)dx = 1, \) for all \( j \geq 1. \)

(1) Let us show that \( \iota \psi_j \to \delta_0 \) in \( \mathcal{D}'(\mathbb{R}). \) By definition, it means that we need to prove that, for all \( \phi \in C_c^\infty(\mathbb{R}), \) \( \int_{\mathbb{R}} \psi_j(x)\phi(x)dx \to \delta_0(\phi) = \phi(0). \) So let’s prove that for every \( \epsilon > 0, \) there exists \( j_0 \) such that for all \( j \geq j_0, \)

\[
\left| \int_{\mathbb{R}} \psi_j(x)\phi(x)dx - \phi(0) \right| < \epsilon.
\]

Let \( \epsilon > 0 \) and \( \phi \in C_c^\infty(\mathbb{R}). \) Since \( \phi \) is continuous, there exists \( \eta > 0 \) such that for all \( |x| < \eta, \) \( |\phi(x) - \phi(0)| < \epsilon. \) Then consider \( j_0 \) such that \( 1/j_0 < \eta. \) Since we know that \( \psi_j(x) = 0 \) for all \( |x| \geq 1/j, \) \( \int_{\mathbb{R}} \psi_j(x)dx = 1, \) and \( \psi_j(x) \geq 0 \) for all \( x, \) for all \( j \geq j_0 \) we have:

\[
\left| \int_{\mathbb{R}} \psi_j(x)\phi(x)dx - \phi(0) \right| = \left| \int_{\mathbb{R}} \psi_j(x)\phi(x)dx - \int_{\mathbb{R}} \psi_j(x)\phi(0)dx \right| \\
\leq \int_{-1/j}^{1/j} \psi_j(x) |\phi(x) - \phi(0)| dx \\
< \epsilon \int_{-1/j}^{1/j} \psi_j(x)dx = \epsilon.
\]

(2) Let \( \phi \in C_c^\infty(\mathbb{R}) \) be such that \( \phi(x) = 1 \) for all \( |x| < 1. \) Then, for \( j \geq 1, \)

\[
\int_{\mathbb{R}} \psi_j(x)^2 \phi(x)dx = \int_{-1/j}^{0} j^2(1 + jx)^2 dx + \int_{0}^{1/j} j^2(1 - jx)^2 dx = \frac{2j}{3}.
\]
Therefore, as \( j \to +\infty \), \( \int_{\mathbb{R}} \psi_j(x)^2 \phi(x)dx \to +\infty \), and \( \left\{ t_{\psi_j^2}(\phi) \right\}_{j=1}^{+\infty} \) does not converge.

Consequently, \( \left\{ t_{\psi_j^2}(\phi) \right\}_{j=1}^{+\infty} \) does not converge to any distribution since \( \left\{ t_{\psi_j^2}(\phi) \right\}_{j=1}^{+\infty} \) does not converge for the very \( \phi \) we exhibited.

(3) We have just shown that \( \psi_j \to \delta_0 \), but \( \left\{ t_{\psi_j^2}(\phi) \right\}_{j=1}^{+\infty} \) does not converge to any distribution. Therefore there is no continuous extension of the map \( Q : f \mapsto f^2 \) on \( C(\mathbb{R}) \) to \( D'(\mathbb{R}) \).

**Problem 2.** We consider the conservation law:

\[ u_t + (f(u))_x = 0, \quad u(x,0) = \phi(x), \]  

with \( f \in C^2(\mathbb{R}) \).

Since \( u \) is continuous and \( f \) is \( C^2 \), \( v = f'(u) \) is also continuous. Since \( u \) is \( C^1 \) apart from jump discontinuities in its first derivatives, away from the jumps, \( u_t \) (resp. \( u_x \)) is perfectly defined and continuous. Therefore, since \( f' \) is \( C^1 \), and away from the discontinuities, \( v_t = f''(u)u_t \) (resp. \( v_x = f''(u)u_x \)) is also continuous, \( v \) is \( C^1 \) apart from jump discontinuities in its first derivatives (the same ones as \( u \)). Therefore \( v \) has the same properties as \( u \). Moreover, we have, away from discontinuities:

\[ v_t + vv_x = f''(u)u_t + f'(u)f''(u)u_x = f''(u)(u_t + f'(u)u_x) = f''(u)(u_t + (f(u))_x) = 0. \]  

(6)

So \( v \) verifies the Burger’s equation (the Rankine-Hugoniot condition is vacuous: there are no shock since \( v \) is continuous).

If \( f'' > 0 \), \( f \) is strictly convex and \( f' \) is strictly increasing and therefore the inverse function \( (f')^{-1} \) exists. We can therefore first solve for \( v \) from the Burger’s equation:

\[ v_t + vv_x = 0, \quad v(x,0) = f'(\phi(x)), \]  

and then \( u = (f')^{-1}(v) \) is solution of the original PDE.

Suppose now that \( u \) has a jump discontinuity. Then, according to the Rankine-Hugoniot condition:

\[ \xi'(t) = \frac{f(u_+)-f(u_-)}{u_+ - u_-}. \]  

(8)

If \( v \) could have been defined as previously, \( v \) would have the same discontinuity \( (v = f'(u)) \) and again by Rankine-Hugoniot:

\[ \xi'(t) = \frac{v_+^2 - v_-^2}{v_+ - v_-} = \frac{1}{2}(v_+ + v_-) = \frac{1}{2}(f'(u_+) + f'(u_-)). \]  

(9)
But in general, 
\[ \frac{f(u_+) - f(u_-)}{u_+ - u_-} \neq \frac{1}{2} (f'(u_+) + f'(u_-)) \]  
(consider for instance \( f(x) = e^x \)) and the statement is FALSE.

**Problem 3.** Consider Burger’s equation 
\[ u_t + uu_x = 0, \quad u(x,0) = \phi(x), \]  
with initial condition 
\[ \phi(x) = \begin{cases} 0, & x < -1, \\ -1 - x, & -1 < x < 0, \\ -1 + x, & 0 < x < 1, \\ 0, & x > 1. \end{cases} \]

(1) To build the weak solution, it is very convenient to draw the characteristic curves.

In the Burger case, we know that the solution \( u \) is constant along the characteristic curves \( x_r(t) = \phi(r)t + r \). As long as the characteristic curves don’t intersect, we have:

- If \( r < -1 \), then \( \phi(r) = 0 \), which implies that the characteristic curves are 
  \[ x_r(t) = r, \quad r < -1, \]  
  and the solution \( u(x,t) = 0 \) along those curves,

- If \( -1 < r < 0 \), then \( \phi(r) = -1 - r \), which implies that the characteristic curves are 
  \[ x_r(t) = (-1 - r)t + r, \quad -1 < r < 0, \]  
  and the solution 
  \[ u(x,t) = -1 - r = -1 - \frac{x + t}{t - 1} = \frac{x + 1}{t - 1} \]  
  along those curves.

- If \( 0 < r < 1 \), then \( \phi(r) = -1 + r \), which implies that the characteristic curves are 
  \[ x_r(t) = (-1 + r)t + r, \quad 0 < r < 1, \]  
  and the solution 
  \[ u(x,t) = -1 + r = -1 + \frac{x + t}{1 + t} = \frac{x - 1}{t + 1} \]  
  along those curves.

- If \( r > 1 \), then \( \phi(r) = 0 \), which implies that the characteristic curves are 
  \[ x_r(t) = r, \quad r > 1, \]  
  and the solution \( u(x,t) = 0 \) along those curves.
To sum up, we have that, for \( t \) small,

\[
 u(x,t) = \begin{cases} 
 0, & x < -1, \\
 \frac{x + 1}{t - 1}, & -1 < x < -t, \quad ( -1 < \frac{x + 1}{t - 1} < 0 ) \\
 \frac{x - 1}{t + 1}, & -1 < x < -t, \quad ( 0 < \frac{x - 1}{t + 1} < 1 ) \\
 0, & x > 1. 
\end{cases}
\] (17)

Now, from the sketch of the characteristic curves and/or the condition \(-1 < x < -t\) of the solution, we can see that the characteristic curves don’t intersect while \( t < 1 \). Therefore the above solution is valid for \( t < 1 \).

The curves intersect at \( t = 1 \) Beyond that time, we therefore consider a weak solution satisfying the Rankine-Hugoniot condition. At the level of the discontinuity \( \xi(t) \), we have that \( u_-(\xi(t),t) = 0 \) and \( u_+(\xi(t),t) = \frac{\xi(t) - 1}{t + 1} \), and \( \xi(1) = -1 \). Therefore the Rankine-Hugoniot condition is

\[
 \xi'(t) = \frac{0 - \frac{1}{2} \left( \frac{\xi(t) - 1}{t + 1} \right)^2}{0 - \frac{\xi(t) - 1}{t + 1}} = \frac{1}{2} \frac{\xi(t) - 1}{t + 1}. \] (18)

Hence, \( \xi \) verifies the following ODE:

\[
 \begin{cases} 
 \xi'(t) = \frac{1}{2} \frac{\xi(t) - 1}{t + 1}, \\
 \xi(1) = -1. 
\end{cases} \] (19)

Solve it (say, by separation of variables) and you get \( \xi(t) = 1 - \sqrt{2(1+t)} \).

Therefore, for \( t \geq 1 \), the solution is

\[
 u(x,t) = \begin{cases} 
 0, & -1 < x < 1 - \sqrt{2(1+t)}, \\
 \frac{x - 1}{t + 1}, & 1 - \sqrt{2(1+t)} < x < 1, \\
 0, & x > 1. 
\end{cases}
\] (20)

(2) There are two cases for \( t \) to consider.

For \( 0 \leq t < 1 \),

\[
 \int_{\mathbb{R}} u(x,t)dx = \int_{-1}^{-t} \frac{x + 1}{t - 1}dx + \int_{-t}^{1} \frac{x - 1}{t + 1}dx = -1. \] (21)

And for \( t > 1 \),

\[
 \int_{\mathbb{R}} u(x,t)dx = \int_{1 - \sqrt{2(1+t)}}^{1} \frac{x - 1}{t + 1}dx = -1. \] (22)

Therefore \( \int_{\mathbb{R}} u(x,t)dx \) is indeed constant.
Let us call $E(t) = \int_{\mathbb{R}} w(x,t)dx$, where $w = u^3$.

For $0 \leq t < 1$, we have

$$E(t) = \int_{\mathbb{R}} w(x,t)dx = \int_{-1}^{-t} (\frac{x+1}{t-1})^3 dx + \int_{-t}^{1} (\frac{x-1}{t+1})^3 dx = -\frac{1}{2}.$$  \hspace{1cm} (23)

So $E(t)$ is indeed constant before a shock develops, but for $t > 1$,

$$E(t) = \int_{\mathbb{R}} w(x,t)dx = \int_{1-\sqrt{2(1+t)}}^{1} (\frac{x-1}{t+1})^3 dx = -\frac{1}{t+1},$$  \hspace{1cm} (24)

and $E(t)$ is no longer a constant.

Let’s now explain what is going on here.

Following Problem 2, if we define $g(x) = \frac{3}{4}x^{4/3}$, we have that $u = g'(w)$.

From Problem 2, we know that $w_t + (g(w))_x = 0$. After the shock develops, we have for $u$:

$$\frac{d}{dt} \int_{\mathbb{R}} u(x,t)dx = \frac{d}{dt} \int_{\xi(t)}^{1} u(x,t)dx$$

$$= \int_{\xi(t)}^{1} \left( u_+ \right)_t - \xi'(t) u_+ (\xi(t),t)$$

$$= \int_{\xi(t)}^{1} \left( \frac{1}{2} u_+^2 \right)_x - \xi'(t) u_+ (\xi(t),t)$$

$$= \frac{1}{2} u_+ (\xi(t),t)^2 - \xi'(t) u_+ (\xi(t),t) = 0,$$  \hspace{1cm} (25)

from the Rankine-Hugoniot jump condition (remember $u_- = 0$).

Meanwhile for $w$ we have:

$$\frac{d}{dt} \int_{\mathbb{R}} w(x,t)dx = \frac{d}{dt} \int_{\xi(t)}^{1} w(x,t)dx$$

$$= \int_{\xi(t)}^{1} \left( w_+ \right)_t - \xi'(t) w_+ (\xi(t),t)$$

$$= \int_{\xi(t)}^{1} \left( g(w_+) \right)_x - \xi'(t) w_+ (\xi(t),t)$$

$$= g(w_+(\xi(t),t)) - \xi'(t) w_+ (\xi(t),t) \neq 0,$$  \hspace{1cm} (26)

because $w$ does not satisfy the same Rankine-Hugoniot condition.

**Problem 4.**
u_{xx} - u_{xy} - 2u_{yy} = 0. \hspace{1cm} (27)

We have \( A = \begin{pmatrix} 1 & -1/2 \\ -1/2 & -2 \end{pmatrix} \). Therefore \( \det(A) = -2 - 1/4 = -9/4 < 0 \), and \( Tr(A) = 1 + 2 = 3 < 0 \). Therefore the eigenvalues of \( A \) are non zero and of opposite signs: Hyperbolic PDE.

u_{xx} - 2u_{xy} + u_{yy} = 0. \hspace{1cm} (28)

We have \( A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \). Therefore \( \det(A) = 0 \). Therefore at least one of the eigenvalues of \( A \) is zero: Degenerate PDE.

u_{xx} + 2u_{xy} + 2u_{yy} = 0. \hspace{1cm} (29)

We have \( A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \). Therefore \( \det(A) = 2 - 1 = 1 > 0 \) and \( Tr(A) = 3 > 0 \). Therefore the eigenvalues of \( A \) are non zero and of the same sign: Elliptic PDE.

Problem 5.

1. Let’s find the general \( C^2 \) solution of the PDE

\[ u_{xx} - u_{xt} - 6u_{tt} = 0, \] \hspace{1cm} (30)

by reducing it to a system of first order PDEs (by the way, this is an elliptic PDE).

We are looking for \( a, b, c, d \) that formally verify:

\[
\partial_{xx} - \partial_{xt} - 6\partial_{tt} = (a\partial_x + b\partial_t)(c\partial_x + d\partial_t) = ac\partial_{xx} + (ad + bc)\partial_{xt} + bd\partial_{tt}. \hspace{1cm} (31)
\]

So we get the (under-determined) system:

\[
\begin{cases}
ac = 1 \\
bd = -6, \\
ad + bc = -1,
\end{cases} \hspace{1cm} (32)
\]

From the first equation, let us simply take \( a = c = 1 \). Then the system reduces to

\[
\begin{cases}
d + b = -1 \\
bd = -6,
\end{cases} \hspace{1cm} (33)
\]

which gives \( b = 2 \) and \( d = -3 \).
Therefore we can write \( u_{xx} - u_{xt} - 6u_{tt} = 0 \) as \( (\partial_x + 2\partial_t)(\partial_x - 3\partial_t)u = 0 \). Now let \( v = (\partial_x - 3\partial_t)u \). Then \( v \) verifies \( v_x + 2v_t = 0 \) (first order linear PDE!). And we know that the solution writes \( v(x,t) = h(t - 2x) \) for some \( h \in C^1 \). Now for \( u \) we have the system:

\[
u_x - 3u_t = h(t - 2x).\quad (34)
\]

Using the method of characteristics, we get the following equations:

\[
\begin{cases}
  x'(s) = 1, & x_r(0) = 0, \\
  t'(s) = -3, & t_r(0) = r, \\
  v'(s) = h(t_r(s) - 2x_r(s)), & v_r(0) = \phi(r),
\end{cases}
\quad (35)
\]

for some function \( \phi \in C^2 \). Therefore we have \( x_r(s) = s, t_r(s) = -3s + r \), and

\[
v'(s) = h(-3s + r - 2s) = h(-5s + r), \quad (36)
\]

and by integrating from \( s = 0 \), we get:

\[
v_r(s) = \int_0^s h(-5s' + r)ds' + \phi(r) = -\frac{1}{5} \int_r^{t+3x} h(y)dy + \phi(r), \quad (37)
\]

after a change of variables. Now, since we have \( s = x \) and \( r = t + 3x \), we finally get:

\[
u(x,t) = \frac{1}{5} \int_{t-2x}^{t+3x} h(y)dy + \phi(t + 3x) = f(t + 3x) + g(t - 2x), \quad (38)
\]

for some \( f, g \in C^2 \).

Reciprocally, we verify that \( u \) of the form \( u(x,t) = f(t + 3x) + g(t - 2x) \) for \( f, g \in C^2 \) indeed solves the PDE.

(2) For an arbitrary \( \phi \in C^\infty_c(\mathbb{R}^2) \) we have to show that

\[
u(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) = v(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) + w(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) = 0. \quad (39)
\]

But from Problem 2 of Pset 2, we have:

\[
v(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) = v((\partial_x - 3\partial_t)(\phi_x + 2\phi_t)) = 0, \quad (40)
\]

and similarly,

\[
w(\phi_{xx} - \phi_{xt} - 6\phi_{tt}) = w((\partial_x + 2\partial_t)(\phi_x - 3\phi_t)) = 0. \quad (41)
\]
Problem 6. Let us solve (in the strong sense):

\[
\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, \begin{cases}
    u_{xx} + 3u_{xy} - 4u_{yy} = xy, \\
    u(x, x) = \sin x, \quad u_x(x, x) = 0.
\end{cases} \tag{42}
\]

Same strategy here, we reduce it to a system of first order PDEs (by the way, this is a hyperbolic PDE!).

We are looking for \(a, b, c, d\) that formally verify:

\[
\partial_{xx} + 3\partial_{xy} - 4\partial_{yy} = (a\partial_x + b\partial_y)(c\partial_x + d\partial_y) = ac\partial_{xx} + (ad + bc)\partial_{xy} + bd\partial_{yy}. \tag{43}
\]

So we get the (under-determined) system:

\[
\begin{cases}
    ac = 1 \\
    ad + bc = 3 \\
    bd = -4.
\end{cases} \tag{44}
\]

From the first equation, let us simply take \(a = c = 1\). Then the system reduces to

\[
\begin{cases}
    d + b = 3 \\
    bd = -4,
\end{cases} \tag{45}
\]

which gives, say, \(b = -1\) and \(d = 4\).

Therefore we can write \(u_{xx} + 3u_{xy} - 4u_{yy} = 0\) as \((\partial_x + 4\partial_y)u = 0\). Now let \(v = (\partial_x + 4\partial_y)u\). Then \(v\) verifies \(v_x - v_y = xy\) (first order semilinear PDE!), and \((u_x + u_y)|_{(x, x)} = \sin'(x) = \cos(x)\), \(u_x(x, x) = 0\) imply that \(u_y(x, x) = \cos(x)\) and \(v(x, x) = 4\cos(x)\). Therefore \(v\) satisfies the following PDE:

\[
\begin{cases}
    v_x - v_y = xy, \\
    v(x, x) = 4\cos(x).
\end{cases} \tag{46}
\]

The ODEs for the characteristics are then:

\[
\begin{cases}
    x'_r(s) = 1, \\
    y'_r(s) = -1, \\
    v'_r(s) = x_r(s)y_r(s), \quad v_r(0) = 4\cos(r).
\end{cases} \tag{47}
\]

After solving, we get:

\[
\begin{cases}
    x_r(s) = s + r, \\
    y_r(s) = r - s, \\
    v_r(s) = v^2s - \frac{s^3}{3} + 4\cos(r).
\end{cases} \tag{48}
\]

Therefore we get the PDE for \(u\):

\[
\begin{cases}
    u_x + 4u_y = v(x, y) = \left(\frac{x + y}{2}\right)^2 \left(\frac{x - y}{2}\right) + \frac{1}{3} \left(\frac{x - y}{2}\right)^3 + 4\cos\left(\frac{x + y}{2}\right), \\
    u(x, x) = \sin(x).
\end{cases} \tag{49}
\]
Writing once again the characteristic ODEs for the PDE, we get:

\[ \begin{align*}
    x'_r(s) &= 1, & x_r(0) &= r, \\
    y'_r(s) &= 4, & y_r(0) &= r, \\
    v'_r(s) &= v(x_r(s), y_r(s)), & v_r(0) &= \sin(r). \\
\end{align*} \tag{50} \]

After solving, we get:

\[ \begin{align*}
    x_r(s) &= s + r, \\
    y_r(s) &= 4s + r, \\
    v_r(s) &= \sin(r) + \int_0^s \left(-\frac{1}{3} \left(-\frac{3}{2} \tau \right)^3 - \left(\frac{5\tau + 2r}{2} \right)^2 \frac{3\tau}{2} + 4\cos \left(\frac{5\tau + 2r}{2} \right) \right) d\tau \\
    &= \frac{9}{32} s^4 - \frac{75}{32} s^4 - \frac{5}{2} s^3 r - \frac{3}{4} s^2 r^2 + \frac{8}{5} \sin \left(\frac{5s + 2r}{2} \right) - \frac{3}{5} \sin (r). \tag{51} \end{align*} \]

Therefore (and finally):

\[ u(x, t) = -\frac{1}{16} \left(\frac{y - x}{3}\right)^2 \left(\frac{5x + y}{3}\right) \left(\frac{5y + 13x}{3}\right) + \frac{8}{5} \sin \left(\frac{x + y}{2}\right) - \frac{3}{5} \sin \left(\frac{4x - y}{3}\right). \tag{52} \]

**Problem 7.** Let’s solve the wave equation on the line:

\[ u_{tt} - c^2 u_{xx} = 0, \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x), \tag{53} \]

with

\[ \phi(x) = \begin{cases} 
    0, & x < -1, \\
    1 + x, & -1 < x < 0, \\
    1 - x, & 0 < x < 1, \\
    0, & x > 1, \end{cases} \tag{54} \]

and

\[ \psi(x) = \begin{cases} 
    0, & x < -1, \\
    2, & -1 < x < 1, \\
    0, & x > 1. \end{cases} \tag{55} \]

We know from the course (method of characteristics) that the solution is:

\[ u(x, t) = \frac{1}{2} \left(\phi(x + ct) + \phi(x - ct)\right) + \frac{1}{2c} \int_{x - ct}^{x + ct} \psi(y) dy. \tag{56} \]

Now, notice first that since \( c, t \geq 0 \), then \( x - ct \leq x + ct \). As you can see in the figure, there are 10 cases to consider for the solution:

1. If \( x + ct < -1 \) and \( x - ct < -1 \) (domain 1), then \( u(x, t) = 0 \).
2. If $-1 \leq x + ct < 0$ and $x - ct < -1$ (domain 2), then
   \[
u(x,t) = \frac{1}{2} (1 + x + ct + 0) + \frac{1}{2c} \int_{-1}^{x+ct} 2dy = \left( \frac{1}{2} + \frac{1}{c} \right) (1 + x + ct). \tag{57}\]

3. If $0 \leq x + ct < 1$ and $x - ct < -1$ (domain 3), then
   \[
u(x,t) = \frac{1}{2} (1 - x - ct + 0) + \frac{1}{2c} \int_{-1}^{x+ct} 2dy = \frac{1}{2} (1 - x - ct) + \frac{1}{c} (1 + x + ct). \tag{58}\]

4. If $1 \leq x + ct$ and $x - ct < -1$ (domain 4), then
   \[
u(x,t) = 0 + \frac{1}{2c} \int_{-1}^{1} 2dy = \frac{2}{c}. \tag{59}\]

5. If $-1 \leq x + ct < 0$ and $-1 < x - ct < 0$ (domain 5), then
   \[
u(x,t) = \frac{1}{2} (1 + x + ct + 1 + x - ct) + \frac{1}{2c} \int_{-ct}^{x+ct} 2dy = 1 + x + 2t. \tag{60}\]

6. If $0 \leq x + ct < 1$ and $-1 < x - ct < 0$ (domain 6), then
   \[
u(x,t) = \frac{1}{2} (1 - x - ct + 1 + x - ct) + \frac{1}{2c} \int_{-ct}^{x+ct} 2dy = 1 - ct + 2t. \tag{61}\]

7. If $1 \leq x + ct$ and $-1 \leq x - ct < 0$ (domain 7), then
   \[
u(x,t) = \frac{1}{2} (0 + 1 + x - ct) + \frac{1}{2c} \int_{-ct}^{1} 2dy = \frac{1}{2} (1 + x - ct) + \frac{1}{c} (1 - x + ct). \tag{62}\]

8. If $0 \leq x + ct < 1$ and $0 \leq x - ct < 1$ (domain 8), then
   \[
u(x,t) = \frac{1}{2} (1 - x - ct + 1 - x + ct) + \frac{1}{2c} \int_{-ct}^{x+ct} 2dy = 1 - x + 2t. \tag{63}\]

9. If $1 \leq x + ct$ and $0 \leq x - ct < 1$ (domain 9), then
   \[
u(x,t) = \frac{1}{2} (1 - x - ct) + \frac{1}{2c} \int_{-ct}^{1} 2dy = \frac{1}{2} (1 + x - ct) + \frac{1}{c} (1 - x + ct). \tag{64}\]

10. If $1 \leq x + ct$ and $1 \leq x - ct$ (domain 10), then $\nu(x,t) = 0$.

Finally, as we can now see, $\nu(x,t)$ vanishes in region (1) and (10). Since $\phi(x) = 0$ and $\psi(x) = 0$ in $|x| > 1$, this result corresponds to Huygens’ principle. Moreover, $\nu(x,t)$ is $C^1$ except on the lines $x + ct = -1$, $x + ct = 0$, $x + ct = 1$, $x - ct = -1$, $x - ct = 0$ and $x - ct = 1$. Since $\phi(x)$ and $\psi(x)$ are $C^1$ everywhere except at $x = -1, 0, 1$, this result corresponds to the propagation of singularities: $\nu(x,t)$ is $C^1$ near $(x,t)$ if $\phi$ and $\psi$ are such near $x \pm ct$. 
