MATH 220: Problem Set 5
Solutions

Problem 1. Let $f \in L^1(\mathbb{R}^n)$ and $a \in \mathbb{R}^n$. The whole problem is a matter of change of variables with integrals.

(i) 
\[ (\mathcal{F}f_a)(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f_a(x) \, dx = \int_{\mathbb{R}^n} e^{-i(x-a) \cdot \xi} f(x-a) \, dx \]
\[ = \int_{\mathbb{R}^n} e^{-i(a+y) \cdot \xi} f(y) \, dy = e^{-ia \cdot \xi} \int_{\mathbb{R}^n} e^{-iy \cdot \xi} f(y) \, dy \]
\[ = e^{-ia \cdot \xi} (\mathcal{F}f_a)(\xi). \tag{1} \]

(ii) 
\[ (\mathcal{F}g_a)(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} g_a(x) \, dx = \int_{\mathbb{R}^n} e^{-i(x-a) \cdot \xi} f(x) \, dx \]
\[ = \int_{\mathbb{R}^n} e^{-i(\xi-a) \cdot x} f(x) \, dx \]
\[ = (\mathcal{F}f)(\xi-a). \tag{2} \]

(iii) 
\[ (\mathcal{F}^{-1}f_a)(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f_a(x) \, dx = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-a) \cdot \xi} f(x-a) \, dx \]
\[ = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(a+y) \cdot \xi} f(y) \, dy = e^{ia \cdot \xi} (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iy \cdot \xi} f(y) \, dy \]
\[ = e^{ia \cdot \xi} (\mathcal{F}^{-1}f_a)(\xi). \tag{3} \]

(iv) 
\[ (\mathcal{F}^{-1}g_a)(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} g_a(x) \, dx = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} e^{ia \cdot x} f(x) \, dx \]
\[ = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(\xi + a) \cdot x} f(x) \, dx \]
\[ = (\mathcal{F}^{-1}f)(\xi + a). \tag{4} \]
Problem 2. Let $f \in C^1(\mathbb{R}^n)$, and $|x|^N f(x)$, $|x|^N \partial_j f(x)$ bounded with $N > n$.

Let’s write $x = (x_1, ..., x_j, ..., x_n)$ and take $h_j = (0, ..., h, ..., 0) = he_j$ (zero everywhere except for the $j^{th}$ coordinate).

We have

$$ (\mathcal{F}(\partial_j f))(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} (\partial_j f)(x) dx = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \lim_{h \to 0} \frac{f(x + h e_j) - f(x)}{h} dx $$

If we can switch limit and integral, we are done. Therefore we need to justify the inversion.

Let’s take $h_j^k = (0, ..., h^k, ..., 0) = h^k e_j$, with $(h^k)_{k \geq 0}$ such that $h^k \to 0$ as $k \to +\infty$. Now we can consider $g_k(x) = \frac{f(x + h_j^k) - f(x)}{h^k}$.

- $g_k$ is integrable for all $k \geq 0$ since $f$ continuous and $|x|^N f(x)$ is bounded,
- $g_k(x) \to \partial_j f(x)$ as $k \to +\infty$ pointwise,
- for all $k \geq 0$, $|g_k(x)| \leq \phi(x)$, with $\phi \in L^1(\mathbb{R}^n)$, $\phi(x) \geq 0$. Indeed, by the mean value theorem, there exists $\eta_j^k \in (x_j^k, x_j^k + h^k)$ such that $g_k(x) = \partial_j f(x_1, ..., \eta_j^k, ..., x_n)$. Now, for instance, for $|x| < 1$, there exists $M_1 > 0$ such that $|\partial_j f(x)| \leq M_1 (\partial_j f$ continuous), and for $|x| > 1$, there exists $M_2 > 0$ such that $|\partial_j f(x)| \leq \frac{M_2}{|x|^N}$ by assumption.

Hence, $|g_k(x)| \leq M_1 \chi_{|x|<1} + \frac{M_1}{|x|^N} \chi_{|x|>1} = \phi(x)$, with $\chi$ the indicator function.

Therefore, we can use the Lebesgue Dominated Convergence Theorem (LDCT) on $h_k(x; \xi) = e^{-ix \cdot \xi} g_k(x)$ (same conclusions since $|e^{-ix \cdot \xi}| = 1$) and write:

$$ (\mathcal{F}(\partial_j f))(\xi) = \int_{\mathbb{R}^n} \lim_{k \to +\infty} h_k(x; \xi) dx $$

$$ = \lim_{k \to +\infty} \int_{\mathbb{R}^n} h_k(x; \xi) dx $$

$$ = \lim_{k \to +\infty} \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \frac{f(x + h_j^k) - f(x)}{h^k} dx $$

$$ = \lim_{k \to +\infty} \frac{e^{ih_j^k \xi} (\mathcal{F} f)(\xi) - (\mathcal{F} f)(\xi)}{h^k} $$

$$ = \left( \lim_{k \to +\infty} \frac{e^{ih_j^k \xi} - 1}{h^k} \right) (\mathcal{F} f)(\xi) $$

$$ = \left( \lim_{k \to +\infty} \frac{e^{ih_j^k \xi} - 1}{h^k} \right) (\mathcal{F} f)(\xi) $$

$$ = i \xi_j (\mathcal{F} f)(\xi) $$

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In short, if $f$ is $C^1$ and $|x|^N f, |x|^N \partial_j f$ are bounded, with $N > n$, we can invert integral and limit. Also, we could have used integration by parts to get to the result.

An alternative of the LDCT is to use the following argumentation.

$(1 + |x|)^N \frac{f(x + h_j) - f(x)}{h}$ converges uniformly on the real line corresponding to the $x_j$-axis to $(1 + |x|)^N \partial_j f(x)$ for $N' < N$. Indeed, the difference quotient is

$$\int_0^1 \partial_j f(x + sh_j)ds$$

by Taylor’s theorem (or simply the fundamental theorem of calculus with a change of variable), and by assumption this integral is bounded by $C(1 + |x|)^{-N}$. So first $(1 + |x|)^N \frac{f(x + h_j) - f(x)}{h}$ is uniformly bounded (in $|h| \leq 1$, $h \neq 0$ and $x \in \mathbb{R}^n$).

Further, it differs from $\partial_j f(x)$ by

$$\int_0^1 (\partial_j f(x + sh) - \partial_j f(x))ds$$

Since $\partial_j f$ is continuous, it is uniformly continuous on compact sets, and since $(1 + |x|)^N \partial_j f$ is bounded, $(1 + |x|)^{N'} \partial_j f$ is uniformly continuous for $N' < N$. (The extra decay factor $(1 + |x|)^{(N' - N)}$ makes sure it goes to 0 at infinity, which is why uniform continuity is easy.) Thus, given $\epsilon > 0$, for sufficiently small $h$, and for $s \in [0, 1]$, one has for all $x \in \mathbb{R}^n$,

$$(1 + |x|)^{N'} |\partial_j f(x + sh_j) - \partial_j f(x)| < \epsilon$$  \hspace{1cm} (7)

Thus, $|\frac{f(x + h_j) - f(x)}{h} - \partial_j f(x)| < \epsilon$ for sufficiently small $h$, and so, with $(1 + |x|)^{-N'}$ being integrable,

$$\int \left| \frac{f(x + h_j) - f(x)}{h} - \partial_j f(x) \right| dx < C' \epsilon,$$  \hspace{1cm} (8)

and thus $\int \frac{f(x + h_j) - f(x)}{h} dx - \int \partial_j f(x) dx < C' \epsilon$ for small $h$. This gives the desired convergence (since you can add the extra multiplicative coefficient $e^{-ix\xi}$ with no significant effect on the conclusions).

Problem 3.

(i) $H(a - |x|) = 1$ if $-a < x < a$, and 0 otherwise. Therefore

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{R}} e^{-ix\xi} H(a - |x|)dx = \int_{-a}^{a} e^{-ix\xi} dx = \frac{e^{ia\xi} - e^{-ia\xi}}{i\xi}.$$  \hspace{1cm} (9)
(ii) \( H(x)e^{-ax} \) is 0 if \( x < 0 \). Therefore:

\[
(\mathcal{F}f)(\xi) = \int_{\mathbb{R}} e^{-ix\xi} e^{-ax} H(x) dx = \int_{0}^{+\infty} e^{-(i\xi+a)x} dx = \frac{1}{i\xi + a}.
\]  

(10)

(iii)

\[
(\mathcal{F}f)(\xi) = \int_{\mathbb{R}} e^{-ix|\xi|n} e^{-a|x|} dx = \int_{0}^{+\infty} x^n e^{-(i\xi+a)x} dx + \int_{-\infty}^{0} (-x)^n e^{-(i\xi-a)x} dx
\]

If we consider the first integral, doing integrations by part \( n \) times, we get:

\[
\int_{0}^{+\infty} x^n e^{-(i\xi+a)x} dx = (-1)^n \frac{n!}{(-a - i\xi)^n} \int_{0}^{+\infty} e^{-(i\xi-a)x} dx
\]

\[
= (-1)^{n+1} \frac{n!}{(-a - i\xi)^{n+1}} = \frac{n!}{(a + i\xi)^{n+1}}
\]  

(12)

Similarly, for the second term, we have

\[
\int_{-\infty}^{0} (-x)^n e^{-(i\xi-a)x} dx = \frac{n!}{(a - i\xi)^{n+1}}
\]  

(13)

Therefore

\[
(\mathcal{F}f)(\xi) = \frac{n!}{(a + i\xi)^{n+1}} + \frac{n!}{(a - i\xi)^{n+1}}.
\]  

(14)

(iv) We can write

\[
\frac{1}{1 + x^2} = \frac{1}{2} \left( \frac{1}{1 + ix} + \frac{1}{1 - ix} \right).
\]  

(15)

Furthermore, we have

\[
\left( \mathcal{F}^{-1}e^{-|x|} \right)(\xi) = \frac{1}{2\pi} \left( \int_{-\infty}^{0} e^{ix\xi}e^{-x} dx + \int_{0}^{+\infty} e^{ix\xi}e^{-x} dx \right)
\]

\[
= \frac{1}{2\pi} \left( - \int_{-\infty}^{0} e^{-iy\xi}e^{-y} dy - \int_{0}^{+\infty} e^{-iy\xi}e^{y} dy \right)
\]

\[
= \frac{1}{2\pi} \left( \int_{0}^{+\infty} e^{-iy\xi}e^{-y} dy + \int_{-\infty}^{0} e^{-iy\xi}e^{y} dy \right)
\]

\[
= \frac{1}{2\pi} \left( \frac{1}{1 + i\xi} + \frac{1}{1 - i\xi} \right).
\]  

(16)
by change of variable and using the result from part (iii) with \( n = 0 \).

Now using the Fourier inversion formula, we deduce that:

\[
\left( \mathcal{F} \frac{1}{1 + x^2} \right) (\xi) = \pi \left( \mathcal{F} \mathcal{F}^{-1} e^{-|x|} \right) (\xi)
= \pi e^{-|\xi|}.
\]  

(17)

Problem 4. We have that

\[
(\mathcal{F} f)(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} |x|^n e^{-a|x|} \, dx.
\]  

(18)

Let \( T \) be a linear transformation such that we rotate the \( z \)-axis to \( \xi \) (we fix \( \xi \in \mathbb{R}^3 \)). Therefore \( T([0,0,1]) = \xi \). Let \( T([1,0,0]) = \xi_1 \) and \( T([0,1,0]) = \xi_2 \).

We can use spherical coordinates such that \( x = (x_1, x_2, x_3) \) into \((r, \theta, \phi)\), where \( r = |x| \), \( \theta \) is the angle between \( \xi \) and \( x \), and \( \phi \) is an angle of rotation about \( \xi \) from \( \xi_2 \).

Since \( x \cdot \xi = |x||\xi| \cos(\theta) \), we obtain that

\[
(\mathcal{F} f)(\xi) = \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=+\infty} \int_{\phi=0}^{\phi=2\pi} e^{-ir|\xi| \cos(\theta)} r^n e^{-ar} r^2 \sin(\theta) \, d\phi \, dr \, d\theta
= 2\pi \int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=+\infty} r^n e^{-ar} r^2 \sin(\theta) \, dr \, d\theta
= 2\pi \int_{\theta=0}^{\theta=\pi} (n + 2)! \left( \frac{1}{(a + i|\xi| \cos(\theta))^{n+3}} - \frac{1}{(a - i|\xi|)^{n+2}} \right) \sin(\theta) \, d\theta
= 2\pi i \int_{\theta=0}^{\theta=\pi} \left( \frac{1}{|\xi|} \frac{1}{(a + i|\xi|)^{n+2}} - \frac{1}{(a - i|\xi|)^{n+2}} \right) \, d\theta
\]  

(19)

Problem 5.

(i) By integration by part, we get

\[
(\mathcal{F}_z D_z f)(y, \zeta) = \frac{1}{i} \int_{\mathbb{R}^k} e^{-iz \cdot \zeta} \partial_z (y, z) \, dz
= -\frac{1}{i} \int_{\mathbb{R}^k} \partial_z (e^{-iz \cdot \zeta}) (y, z) \, dz
= \zeta_j \int_{\mathbb{R}^k} e^{-iz \cdot \zeta} f(y, z) \, dz
= \zeta_j (\mathcal{F}_z f)(y, \zeta)
\]  

(20)
(ii) By definition, we have that

\[ (F_z D_{y_j} f)(y, \zeta) = \frac{1}{i} \int_{\mathbb{R}^k} e^{-iz \cdot \zeta} \partial_{y_j} f(y, z) dz \]  \hspace{1cm} (21) \]

Now since \( f \) is \( C^1 \), and \( |z|^K f, |z|^K \partial_x f \) are bounded with \( K > k \), using the same argumentation as in Pb2, we can switch the derivative and the integral sign to finally obtain

\[ (F_z D_{y_j} f)(y, \zeta) = D_{y_j} \int_{\mathbb{R}^k} e^{-iz \cdot \zeta} f(y, z) dz = (D_{y_j} F_z f)(y, \zeta). \] \hspace{1cm} (22) \]