4 Green's Functions

In this section, we are interested in solving the following problem. Let Ω be an open, bounded subset of $\mathbb{R}^n$. Consider

$$\begin{cases} -\Delta u = f & x \in \Omega \subset \mathbb{R}^n \\ u = g & x \in \partial\Omega. \end{cases} \tag{4.1}$$

4.1 Motivation for Green's Functions

Suppose we can solve the problem,

$$\begin{cases} -\Delta_y G(x,y) = \delta_x & y \in \Omega \\ G(x,y) = 0 & y \in \partial\Omega \end{cases} \tag{4.2}$$

for each $x \in \Omega$. Then, *formally*, we can say that for u a solution of (4.1),

$$\begin{aligned} u(x) &= \int_\Omega \delta_x u(y)\, dy \\ &= -\int_\Omega \Delta_y G(x,y) u(y)\, dy \\ &= \int_\Omega \nabla_y G(x,y) \cdot \nabla_y u(y)\, dy - \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x,y) u(y)\, dS(y) \\ &= -\int_\Omega G(x,y)\Delta_y u(y)\, dy + \int_{\partial\Omega} G(x,y) \frac{\partial u}{\partial \nu}(y)\, dS(y) - \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x,y) u(y)\, dS(y) \\ &= \int_\Omega G(x,y) f(y)\, dy - \int_{\partial\Omega} \frac{\partial G}{\partial \nu}(y) g(y)\, dS(y). \end{aligned}$$

Now, we do know that the fundamental solution of Laplace's equation $\Phi(y)$ satisfies

$$-\Delta_y \Phi(y) = \delta_0$$

and, moreover,

$$-\Delta_y \Phi(x-y) = \delta_x.$$

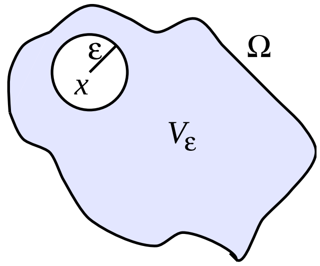
Of course, $\Phi(x-y)$ does not satisfy our boundary conditions, but we will use that as a starting ground to try and construct a solution of (4.2), and, ultimately (4.1). Below, we will also make the formal argument given above more precise.

Recalling the definition of distributional derivative, we will start by looking at

$$\int_\Omega \Phi(x-y)\Delta_y u(y)\, dy.$$

We would like to integrate this term by parts. However, we know that $\Phi(x-y)$ has a singularity at $y = x$. Therefore, in order to integrate by parts, we must proceed as follows.

Fix $x \in \Omega$ and $\epsilon > 0$ such that $\text{dist}(x, \partial\Omega) < \epsilon$ and therefore, $B(x,\epsilon) \subset \Omega$. Let $V_\epsilon \equiv \Omega - B(x,\epsilon)$.



Let Φ be the fundamental solution of Laplace's equation. That is,

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\ln|x| & n=2 \\ \frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}} & n \geq 3. \end{cases}$$

Suppose $u \in C^2(\overline{\Omega})$. By the Divergence Theorem, we have

$$\begin{aligned} \int_{V_\epsilon} \Phi(y-x)\Delta u(y)\,dy &= -\int_{V_\epsilon} \nabla_y \Phi(y-x)\cdot\nabla_y u(y)\,dy + \int_{\partial V_\epsilon} \Phi(y-x)\frac{\partial u}{\partial\nu}\,dS(y) \\ &= \int_{V_\epsilon} \Delta_y\Phi(y-x)u(y)\,dy - \int_{\partial V_\epsilon}\frac{\partial\Phi}{\partial\nu}(y-x)u(y)\,dS(y) \\ &\quad + \int_{\partial V_\epsilon}\Phi(y-x)\frac{\partial u}{\partial\nu}\,dS(y). \end{aligned}$$

where $\frac{\partial u}{\partial\nu}$ denotes the derivative of u in the outer normal direction to V_ϵ. Now on V_ϵ, $\Delta_y\Phi(y-x)=0$. Therefore,

$$\int_{V_\epsilon} \Phi(y-x)\Delta u(y)\,dy = -\int_{\partial V_\epsilon}\frac{\partial\Phi}{\partial\nu}(y-x)u(y)\,dS(y) + \int_{\partial V_\epsilon}\Phi(y-x)\frac{\partial u}{\partial\nu}\,dS(y).$$

Now, we note that

$$\lim_{\epsilon\to 0^+}\int_{V_\epsilon}\Phi(y-x)\Delta u(y)\,dy = \int_\Omega \Phi(y-x)\Delta u(y)\,dy.$$

We make the following claims about the limits of the other two terms as $\epsilon \to 0^+$.

Claim 1.

$$\lim_{\epsilon\to 0^+}\left[-\int_{\partial V_\epsilon}\frac{\partial\Phi}{\partial\nu}(y-x)u(y)\,dS(y)\right] = -\int_{\partial\Omega}\frac{\partial\Phi}{\partial\nu}(y-x)u(y)\,dS(y) - u(x).$$

Claim 2.

$$\lim_{\epsilon\to 0^+}\int_{\partial V_\epsilon}\Phi(y-x)\frac{\partial u}{\partial\nu}(y)\,dS(y) = \int_{\partial\Omega}\Phi(y-x)\frac{\partial u}{\partial\nu}\,dS(y).$$

Assuming these claims for a moment, we conclude that for any $u \in C^2(\overline{\Omega})$,

$$\boxed{u(x) = \int_{\partial\Omega}\left[\Phi(y-x)\frac{\partial u}{\partial\nu}(y) - \frac{\partial\Phi}{\partial\nu}(y-x)u(y)\right]\,dS(y) - \int_\Omega \Phi(y-x)\Delta u(y)\,dy.} \tag{4.3}$$

We would now like to use the representation formula (4.3) to solve (4.1). If we knew Δu on Ω *and* u on $\partial\Omega$ *and* $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$, then we could solve for u. But, we don't know all this information. We know Δu on Ω and u on $\partial\Omega$.

We proceed as follows. For each $x \in \Omega$, we introduce a **corrector function** $h^x(y)$ which satisfies the following boundary-value problem,

$$\begin{cases} \Delta_y h^x(y) = 0 & y \in \Omega \\ h^x(y) = \Phi(y - x) & y \in \partial\Omega. \end{cases} \tag{4.4}$$

Now suppose we can find such a (smooth) function h^x which satisfies (4.4). Then using the same analysis as above, we have

$$\begin{aligned} \int_\Omega h^x(y)\Delta u(y)\,dy &= -\int_\Omega \nabla_y h^x(y) \cdot \nabla_y u(y)\,dy + \int_{\partial\Omega} h^x(y)\frac{\partial u}{\partial \nu}(y)\,dS(y) \\ &= \int_\Omega \Delta_y h^x(y) u(y)\,dy - \int_{\partial\Omega} \frac{\partial h^x}{\partial \nu}(y)u(y)\,dS(y) \\ &\quad + \int_{\partial\Omega} h^x(y)\frac{\partial u}{\partial \nu}(y)\,dS(y). \end{aligned}$$

Now using the fact that h^x is a solution of (4.4), we conclude that

$$\boxed{0 = \int_{\partial\Omega} \left[\Phi(y - x)\frac{\partial u}{\partial \nu}(y) - \frac{\partial h^x}{\partial \nu}(y)u(y)\right] dS(y) - \int_\Omega h^x(y)\Delta u(y)\,dy.} \tag{4.5}$$

Now subtracting (4.5) from (4.3), we conclude that

$$u(x) = -\int_{\partial\Omega} \left[\frac{\partial \Phi}{\partial \nu}(y - x) - \frac{\partial h^x}{\partial \nu}(y)\right] u(y)\,dS(y) - \int_\Omega [\Phi(y - x) - h^x(y)]\Delta u(y)\,dy.$$

Let

$$G(x, y) = \Phi(y - x) - h^x(y).$$

Then, u can be written as

$$\boxed{u(x) = -\int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x, y)u(y)\,dS(y) - \int_\Omega G(x, y)\Delta u(y)\,dy.} \tag{4.6}$$

We define this function G as the **Green's function** for Ω. That is, the Green's function for a domain $\Omega \subset \mathbb{R}^n$ is the function defined as

$$G(x, y) = \Phi(y - x) - h^x(y) \qquad x, y \in \Omega, x \neq y,$$

where Φ is the fundamental solution of Laplace's equation and for each $x \in \Omega$, h^x is a solution of (4.5). We leave it as an exercise to verify that $G(x, y)$ satisfies (4.2) in the sense of distributions.

Conclusion: If u is a (smooth) solution of (4.1) and $G(x, y)$ is the Green's function for Ω, then

$$\boxed{u(x) = -\int_{\partial\Omega} \frac{\partial G}{\partial \nu}(x, y)g(y)\,dS(y) + \int_\Omega G(x, y)f(y)\,dy.} \tag{4.7}$$

We will show below that *conversely* a function of the form (4.7) will give us a solution of (4.1). First, however, we prove the two claims given above.

Proof of Claim 1.

$$-\int_{\partial V_\epsilon} \frac{\partial \Phi}{\partial \nu}(y-x)u(y)\,dS(y) = -\int_{\partial\Omega} \frac{\partial \Phi}{\partial \nu}(y-x)u(y)\,dS(y) + \int_{\partial B(x,\epsilon)} \frac{\partial \Phi}{\partial \nu}(y-x)u(y)\,dS(y).$$

Now

$$\nabla_y \Phi(y) = -\frac{1}{n\alpha(n)}\frac{y}{|y|^n}$$

and the outward normal on $B(x,\epsilon)$ is

$$\nu = \frac{y-x}{|y-x|}.$$

Therefore,

$$\begin{aligned}\frac{\partial \Phi}{\partial \nu}(y-x) &= \nabla_y \Phi(y-x)\cdot \nu \\ &= -\frac{1}{n\alpha(n)}\frac{y-x}{|y-x|^n}\cdot\frac{y-x}{|y-x|} \\ &= -\frac{1}{n\alpha(n)}\cdot\frac{1}{|y-x|^{n-1}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\partial B(x,\epsilon)} \frac{\partial \Phi}{\partial \nu}(y-x)u(y)\,dS(y) &= -\frac{1}{n\alpha(n)}\int_{\partial B(x,\epsilon)} \frac{1}{|y-x|^{n-1}}u(y)\,dS(y) \\ &= -\frac{1}{n\alpha(n)\epsilon^{n-1}}\int_{\partial B(x,\epsilon)} u(y)\,dS(y) \\ &= -⨍_{\partial B(x,\epsilon)} u(y)\,dS(y).\end{aligned}$$

As $\epsilon \to 0^+$,

$$⨍_{\partial B(x,\epsilon)} u(y)\,dS(y) \to u(x).$$

Therefore, we have

$$\lim_{\epsilon\to 0^+}\int_{\partial B(x,\epsilon)} \frac{\partial \Phi}{\partial \nu}(y-x)u(y)\,dS(y) = -u(x),$$

and the claim follows. □

Proof of Claim 2.

Now we know

$$\int_{\partial V_\epsilon} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) = \int_{\partial\Omega} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) - \int_{\partial B(x,\epsilon)} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y).$$

We just need to show that

$$\int_{\partial B(x,\epsilon)} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) \to 0 \qquad \text{as } \epsilon \to 0^+.$$

Substituting in the explicit formula for Φ for $n \geq 3$ (the case $n = 2$ can be handled similarly), we see that

$$\begin{aligned}
\left|\int_{\partial B(x,\epsilon)} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y)\right| &\leq \frac{1}{n\alpha(n)(n-2)}\int_{\partial B(x,\epsilon)} \frac{1}{|y-x|^{n-2}}\left|\frac{\partial u}{\partial \nu}(y)\right|\,dS(y) \\
&\leq \left|\frac{\partial u}{\partial \nu}\right|_{L^\infty(B(x,\epsilon))} \frac{1}{n\alpha(n)\epsilon^{n-2}}\int_{\partial B(x,\epsilon)} dS(y) \\
&\leq C\epsilon ⨍_{\partial B(x,\epsilon)} dS(y) \\
&= C\epsilon.
\end{aligned}$$

Therefore, as $\epsilon \to 0^+$,

$$\int_{\partial B(x,\epsilon)} \Phi(y-x)\frac{\partial u}{\partial \nu}(y)\,dS(y) \to 0$$

as claimed. Therefore, the claim follows. □

Above we have proven the following theorem.

Theorem 3. *If $u \in C^2(\overline{\Omega})$ is a solution of*

$$\begin{cases} -\Delta u = f & x \in \Omega \subset \mathbb{R}^n \\ u = g & x \in \partial\Omega, \end{cases}$$

where f and g are continuous, then

$$\boxed{u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu}(x,y)\,dS(y) + \int_\Omega f(y)G(x,y)\,dy} \tag{4.8}$$

for $x \in \Omega$, where $G(x,y)$ is the Green's function for Ω.

Corollary 4. *If u is harmonic in Ω and $u = g$ on $\partial\Omega$, then*

$$u(x) = -\int_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu}(x,y)\,dS(y).$$

4.2 Finding Green's Functions

Finding a Green's function is difficult. However, for certain domains Ω with special geometries, it is possible to find Green's functions. We show some examples below.

Example 5. Let $\mathbb{R}^2_+$ be the upper half-plane in $\mathbb{R}^2$. That is, let

$$\mathbb{R}^2_+ \equiv \{(x_1,x_2) \in \mathbb{R}^2 : x_2 > 0\}.$$

We will look for the Green's function for $\mathbb{R}^2_+$. In particular, we need to find a corrector function h^x for each $x \in \mathbb{R}^2_+$, such that

$$\begin{cases} \Delta_y h^x(y) = 0 & y \in \mathbb{R}^2_+ \\ h^x(y) = \Phi(y - x) & y \in \partial\mathbb{R}^2_+. \end{cases}$$

Fix $x \in \mathbb{R}^2_+$. We know $\Delta_y \Phi(y - x) = 0$ for all $y \neq x$. Therefore, if we choose $z \notin \Omega$, then $\Delta_y \Phi(y - z) = 0$ for all $y \in \Omega$. Now, if we choose $z = z(x)$ appropriately, $z \notin \Omega$, such that $\Phi(y - z) = \Phi(y - x)$ for $y \in \partial\Omega$, then letting $h^x(y) = \Phi(y - z(x))$, we will have found a corrector function. Recall that for $n = 2$,

$$\Phi(y - z) = -\frac{1}{2\pi} \ln |y - z|.$$

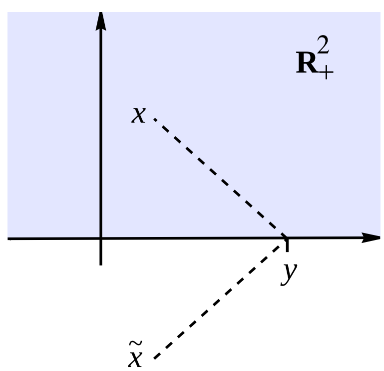
Therefore, $\Phi(y - z)$ is a function of $|y - z|$. Consequently, for $x = (x_1, x_2) \in \mathbb{R}^2_+$, we see that for all $y \in \partial\mathbb{R}^2_+$,

$$|y - x| = |(y_1, 0) - (x_1, x_2)| = |(y_1, 0) - (x_1, -x_2)| = |y - \widetilde{x}|$$

where

$$\widetilde{x} \equiv (x_1, -x_2)$$

is the **reflection of x** in the plane.



Therefore, letting $h^x(y) = \Phi(y - \widetilde{x})$, we have found a corrector function for $\mathbb{R}^2_+$. Therefore, a Green's function for the upper half-plane is given by

$$\begin{aligned} G(x, y) &= \Phi(y - x) - \Phi(y - \widetilde{x}) \\ &= -\frac{1}{2\pi} \left[\ln |y - x| - \ln |y - \widetilde{x}|\right]. \end{aligned}$$

◇

Example 6. More generally, for the upper half-space in $\mathbb{R}^n$,

$$\mathbb{R}^n_+ \equiv \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\},$$

the corrector function $h^x(y)$ is given by

$$h^x(y) = \Phi(y - \widetilde{x})$$

where

$$\widetilde{x} \equiv (x_1, \ldots, x_{n-1}, -x_n)$$

is the **reflection of x** in the plane. Therefore, a Green's function for the upper half-space $\mathbb{R}^n_+$ is given by

$$G(x, y) = \Phi(y - x) - \Phi(y - \widetilde{x}).$$

◇

Example 7. Let $B_2(0,1)$ be the unit ball in $\mathbb{R}^2$. That is, let

$$B_2(0,1) \equiv \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}.$$

Fix $x \in B_2(0,1)$. We need to find a corrector function h^x for $B_2(0,1)$. Again,

$$\Phi(y - x) = -\frac{1}{2\pi} \ln |y - x|.$$

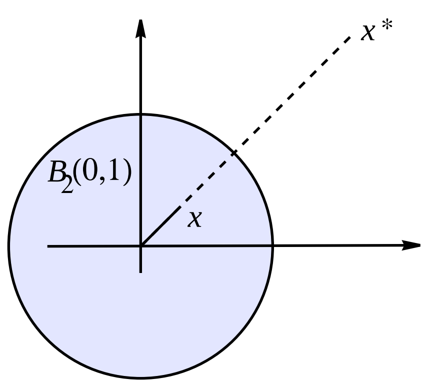
Therefore, $\Phi(y - x)$ is a function of $|y - x|$. We need $h^x(y) = \Phi(y - x)$ for all $y \in \partial B_2(0,1)$, that is, all y such that $|y| = 1$. Now for $y \in \partial B_2(0,1)$,

$$\begin{aligned}
|y - x|^2 &= (y - x) \cdot (y - x) \\
&= |y|^2 - 2y \cdot x + |x|^2 \\
&= |x|^2 - 2x \cdot y + 1 \\
&= |x|^2 |y|^2 - 2x \cdot y + 1 \\
&= |x|^2 \left(|y|^2 - \frac{2x \cdot y}{|x|^2} + \frac{1}{|x|^2} \right) \\
&= |x|^2 \left(|y|^2 - 2y \cdot \frac{x}{|x|^2} + \frac{|x|^2}{|x|^4} \right) \\
&= |x|^2 |y - x^*|^2,
\end{aligned}$$

where

$$x^* = \frac{x}{|x|^2}$$

is called the **point dual to x**.



Notice that for $x \in B_2(0,1)$, x^* is *not* in $B_2(0,1)$. Consequently, we can conclude that $\Phi(|x|(y-x^*))$ is harmonic for all y in Ω. In addition, $\Phi(|x|(y-x^*)) = \Phi(y-x)$ for all $y \in \partial B_2(0,1)$. Therefore, letting

$$h^x(y) = \Phi(|x|(y-x^*)),$$

we see that h^x is a corrector function for the unit ball $B_2(0,1)$.

Consequently, the Green's function for $B_2(0,1)$ is given by

$$\begin{aligned} G(x,y) &= \Phi(y-x) - \Phi(|x|(y-x^*)) \\ &= -\frac{1}{2\pi}\left[\ln|y-x| - \ln[|x||y-x^*|]\right]. \end{aligned}$$

◇

Example 8. For the unit ball in $\mathbb{R}^n$,

$$B_n(0,1) \equiv \{(x_1,\ldots,x_n) : x_1^2 + \ldots + x_n^2 = 1\},$$

a corrector function h^x is given by

$$h^x(y) = \Phi(|x|(y-x^*))$$

where

$$x^* = \frac{x}{|x|^2}$$

is the **point dual to x**. Therefore, a Green's function for $B_n(0,1)$ is given by

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-x^*)).$$

◇

4.3 Using Green's Functions to Solve Poisson's Equation

We have shown above that *if* u is a smooth solution of the Dirichlet problem

$$\begin{cases} -\Delta u = f & x \in \Omega \\ u = g & x \in \partial\Omega, \end{cases}$$

then u can be represented in terms of the Green's function for Ω by (4.8). It remains to show the converse. That is, it remains to show that for continuous functions f, g and a given domain $\Omega \subset \mathbb{R}^n$, the representation formula (4.8) does give us a solution of the Dirichlet problem. First, however, we will use the representation formula (4.8) to write the proposed formula for the solution in the cases above, where we can explicitly calculate the Green's function for the domain Ω.

Example 9. Let $\mathbb{R}^n_+$ be the upper half-space in $\mathbb{R}^n$,

$$\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^2 : x_n > 0\}.$$

From above, we calculated that

$$G(x, y) = \Phi(y - x) - \Phi(y - \widetilde{x})$$

is a Green's function for $\mathbb{R}^n_+$, where $\widetilde{x} = (x_1, \dots, x_{n-1}, -x_n)$ and Φ is the fundamental solution of Laplace's equation in $\mathbb{R}^n$. Now, from (4.8), our proposed solution has the form

$$u(x) = -\int_{\partial \mathbb{R}^n_+} g(y) \frac{\partial G}{\partial \nu}(x, y)\, dS(y) + \int_{\mathbb{R}^n_+} f(y) G(x, y)\, dy.$$

(Note: The analysis in Section 4.1 was carried out under the assumption that Ω was a bounded domain. However, for now we will assume we can use the same representation formula to derive a solution for the unbounded half-space. Later, we will need to prove that this representation formula actually gives us a solution.)

Now, we calculate $\frac{\partial G}{\partial \nu}$ on $\{y_n = 0\}$ to find an explicit formula for solutions to

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ u = g & x \in \partial\Omega. \end{cases}$$

Now

$$\frac{\partial \Phi}{\partial y_n}(y) = -\frac{y_n}{n\alpha(n)|y|^n}.$$

Therefore, the normal derivative of G on $\{y_n = 0\}$ is given by

$$\begin{aligned} \frac{\partial G}{\partial \nu}(x, y) &= \frac{\partial \Phi}{\partial y_n}(y - x) - \frac{\partial \Phi}{\partial y_n}(y - \tilde{x}) \\ &= \frac{y_n - x_n}{n\alpha(n)|y - x|^n} - \frac{y_n - \tilde{x}_n}{n\alpha(n)|y - \tilde{x}|^n} \\ &= -\frac{2x_n}{n\alpha(n)|y - x|^n}. \end{aligned}$$

Therefore, *if* u is the solution of Laplace's equation on the upper half-space Ω with Dirichlet boundary conditions, then we suspect that u will have the form

$$\boxed{u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial R^n_+} \frac{g(y)}{|y - x|^n}\, dy.} \tag{4.9}$$

This is called **Poisson's formula** for the half-space $\mathbb{R}^n_+$. The function

$$K(x, y) = \frac{2x_n}{n\alpha(n)} \frac{1}{|x - y|^n}$$

is called **Poisson's kernel** for the half-space $\mathbb{R}^n_+$.

◇

Example 10. Let $B_n(0,1)$ be the unit ball in $\mathbb{R}^n$. We look for a formula for the solution of Laplace's equation in $B_n(0,1)$ with Dirichlet boundary conditions,

$$\begin{cases} \Delta u = 0 & x \in B_n(0,1) \\ u = g & x \in \partial B_n(0,1). \end{cases} \tag{4.10}$$

By (4.8), if u is a solution of (4.10), then u will have the form

$$u(x) = -\int_{\partial B_n(0,1)} g(y) \frac{\partial G}{\partial \nu}(x,y)\, dS(y).$$

Now we just need to calculate $\frac{\partial G}{\partial \nu}$ on $\partial B_n(0,1)$ where G is a Green's function for $B_n(0,1)$.

As shown above,

$$G(x,y) = \Phi(y-x) - \Phi(|x|(y-x^*))$$

is a Green's function for the unit ball in $\mathbb{R}^n$ where

$$x^* = \frac{x}{|x|^2}$$

is the point dual to x. We consider the case when $n \geq 3$. The case $n = 2$ can be handled similarly. For $n \geq 3$, we have

$$\Phi(y) = \frac{1}{n\alpha(n)} \frac{1}{|y|^{n-2}},$$

which implies

$$\nabla \Phi(y) = -\frac{y}{n\alpha(n)|y|^n}.$$

Therefore,

$$\nabla_y \Phi(y-x) = -\frac{y-x}{n\alpha(n)|y-x|^n},$$

while

$$\begin{aligned} \Phi(|x|(y-x^*)) &= \frac{1}{n\alpha(n)} \frac{1}{||x|(y-x^*)|^{n-2}} \\ &= \frac{1}{|x|^{n-2}} \Phi(y-x^*). \end{aligned}$$

Therefore,

$$\begin{aligned} \nabla_y \Phi(|x|(y-x^*)) &= -\frac{1}{|x|^{n-2}} \frac{y-x^*}{n\alpha(n)|y-x^*|^n} \\ &= -\frac{y|x|^2 - x}{n\alpha(n)||x|(y-x^*)|^n} \\ &= -\frac{y|x|^2 - x}{n\alpha(n)|y-x|^n}. \end{aligned}$$

Now, the unit normal to $B_n(0,1)$ is given by

$$\nu = \frac{y}{|y|} = y.$$

Therefore, the normal derivative of $G(x,\cdot)$ on $\partial B_n(0,1)$ is given by

$$
\begin{aligned}
\frac{\partial G}{\partial \nu}(x,y) &= \frac{\partial \Phi}{\partial \nu}(y-x) - \frac{\partial \Phi}{\partial \nu}(|x|(y-x^*)) \\
&= -\frac{y-x}{n\alpha(n)|y-x|^n}\cdot y + \frac{y|x|^2-x}{n\alpha(n)|y-x|^n}\cdot y \\
&= \frac{-|y|^2 + x\cdot y + |y|^2|x|^2 - x\cdot y}{n\alpha(n)|y-x|^n} \\
&= \frac{|y|^2(|x|^2-1)}{n\alpha(n)|y-x|^n} \\
&= \frac{|x|^2-1}{n\alpha(n)|y-x|^n}.
\end{aligned}
$$

Therefore, the solution formula for (4.10) is given by

$$
\begin{aligned}
u(x) &= -\int_{\partial B_n(0,1)} g(y)\frac{\partial G}{\partial \nu}(x,y)\,dS(y) \\
&= \frac{1-|x|^2}{n\alpha(n)}\int_{\partial B_n(0,1)} \frac{g(y)}{|y-x|^n}\,dS(y).
\end{aligned}
$$

We can use this formula to derive the solution formula for Laplace's equation on the ball of radius r with Dirichlet boundary conditions,

$$
\begin{cases} \Delta u = 0 & x \in B_n(0,r) \\ u = g & x \in \partial B_n(0,r). \end{cases} \tag{4.11}
$$

Suppose u is a solution of (4.11), then $\widetilde{u}(x) = u(rx)$ is a solution of (4.10) with boundary data $\widetilde{g}(x) = g(rx)$. Therefore, by our work above, we see the formula for $\widetilde{u}$ is given by

$$
\begin{aligned}
\widetilde{u}(x) &= \frac{1-|x|^2}{n\alpha(n)}\int_{\partial B_n(0,1)} \frac{\widetilde{g}(y)}{|y-x|^n}\,dS(y) \\
&= (1-|x|^2)\fint_{\partial B_n(0,1)} \frac{g(ry)}{|y-x|^n}\,dS(y) \\
&= (1-|x|^2)\fint_{\partial B_n(0,r)} \frac{g(\widetilde{y})}{|\widetilde{y}/r-x|^n}\,dS(\widetilde{y}) \\
&= r^n(1-|x|^2)\fint_{\partial B_n(0,r)} \frac{g(\widetilde{y})}{|\widetilde{y}-rx|^n}\,dS(\widetilde{y}) \\
&= \frac{r^2-|rx|^2}{n\alpha(n)r}\int_{\partial B_n(0,r)} \frac{g(\widetilde{y})}{|\widetilde{y}-rx|^n}\,dS(\widetilde{y}).
\end{aligned}
$$

Therefore,

$$
u(rx) = \frac{r^2-|rx|^2}{n\alpha(n)r}\int_{\partial B_n(0,r)} \frac{g(y)}{|y-rx|^n}\,dS(y),
$$

which implies the solution formula for (4.11) is given by

$$\boxed{u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B_n(0,r)} \frac{g(y)}{|y-x|^n}\, dS(y).} \tag{4.12}$$

This representation formula is called **Poisson's formula for the ball**. The function

$$\boxed{K(x,y) \equiv \frac{r^2 - |x|^2}{n\alpha(n)r|y-x|^n}} \tag{4.13}$$

is called **Poisson's kernel for the ball**.

◇

As mentioned above, formulas (4.9) and (4.12) will give solutions of

$$\begin{cases} \Delta u = 0 & x \in \Omega \\ u = g & x \in \partial\Omega \end{cases}$$

in the cases when Ω is the half-space and the ball respectively, *assuming* a solution exists. We now will show that these formulas actually solve Laplace's equation with Dirichlet boundary conditions, $u = g$ for $x \in \partial\Omega$.

In particular, we state the following theorems, *(Ref: Evans, Sec. 2.2)*,

Theorem 11. *For $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$ and u defined by (4.9), u satisfies the following,*

1. $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$
2. $\Delta u = 0$ *for* $x \in \mathbb{R}^n_+$
3. $\lim_{\substack{x \to x_0 \\ x \in \mathbb{R}^n_+}} u(x) = g(x_0)$ *for all* $x_0 \in \partial\mathbb{R}^n_+$.

Theorem 12. *For $g \in C(\partial B_n(0,r))$ and u defined by (4.12), u satisfies the following,*

1. $u \in C^\infty(B_n(0,r))$
2. $\Delta u = 0$ *for* $x \in B_n(0,r)$
3. $\lim_{\substack{x \to x_0 \\ x \in B_n(0,r)}} u(x) = g(x_0)$ *for all* $x_0 \in \partial B_n(0,r)$.

We will prove the first of these theorems. The second one follows similarly. In order to prove this theorem, we make use of the following lemma.

Lemma 13. *Green's functions are symmetric. For all $x, y \in \Omega$, $x \neq y$,*

$$G(x,y) = G(y,x).$$

Proof. (Ref: Evans, Sec. 2.2) Fix $x, y \in \Omega$, $x \neq y$. Let

$$\begin{aligned} v(z) &\equiv G(x,z) \\ w(z) &\equiv G(y,z). \end{aligned}$$

We will show that $v(y) = w(x)$, and, therefore, $G(x,y) = G(y,x)$. Recall that

$$G(x,y) = \Phi(y-x) - h^x(y)$$

where $h^x(y)$ satisfies

$$\begin{cases} \Delta_y h^x(y) = 0 & y \in \Omega \\ h^x(y) = \Phi(y-x) & y \in \partial\Omega. \end{cases}$$

Therefore, for $z \in \partial\Omega$,

$$\begin{aligned} v(z) = G(x,z) = \Phi(z-x) - h^x(z) = \Phi(z-x) - \Phi(z-x) = 0 \\ w(z) = G(y,z) = \Phi(z-y) - h^y(z) = \Phi(z-y) - \Phi(z-y) = 0. \end{aligned}$$

Further, $\Delta_z v = 0$ for $z \neq x$ and $\Delta_z w = 0$ for $z \neq y$. Now v is smooth, except near $z = x$, while w is smooth, except near $z = y$. Define the region $V_\epsilon = \Omega - [B(x,\epsilon) \cup B(y,\epsilon)]$ for $\epsilon > 0$. On V_ϵ, our functions are smooth. Therefore, we can use our integration by parts formula as follows,

$$\begin{aligned} \int_{V_\epsilon} \Delta v w \, dz &= -\int_{V_\epsilon} \nabla v \cdot \nabla w \, dz + \int_{\partial V_\epsilon} \frac{\partial v}{\partial \nu} w \, dS(z) \\ &= \int_{V_\epsilon} v \Delta w \, dz - \int_{\partial V_\epsilon} v \frac{\partial w}{\partial \nu} \, dS(z) + \int_{\partial V_\epsilon} \frac{\partial v}{\partial \nu} w \, dS(z) \end{aligned}$$

Using the fact that $\Delta v = 0 = \Delta w$ on V_ϵ, we conclude that

$$\int_{\partial V_\epsilon} v \frac{\partial w}{\partial \nu} \, dS(z) = \int_{\partial V_\epsilon} \frac{\partial v}{\partial \nu} w \, dS(z).$$

Using the fact that $v = 0 = w$ on $\partial\Omega$, we conclude that

$$\int_{\partial B(x,\epsilon)} \left[\frac{\partial v}{\partial \nu} w - \frac{\partial w}{\partial \nu} v \right] \, dS(z) = \int_{\partial B(y,\epsilon)} \left[\frac{\partial w}{\partial \nu} v - \frac{\partial v}{\partial \nu} w \right] \, dS(z)$$

where ν denotes the inward pointing unit vector field on $\partial B(x,\epsilon) \cup \partial B(y,\epsilon)$. Now we claim that as $\epsilon \to 0^+$, the left-hand side converges to $w(x)$, while the right-hand side converges to $v(y)$. For the terms on the left-hand side, we first look at

$$\int_{\partial B(x,\epsilon)} \frac{\partial w}{\partial \nu} v \, dS(z).$$

Now w is smooth near x. Therefore, $\frac{\partial w}{\partial \nu}$ is bounded near $\partial B(x,\epsilon)$. Now $v(z) = \Phi(z-x) - h^x(z)$. Therefore, on $\partial B(x,\epsilon)$, $v(z) \approx 1/\epsilon^{n-2}$. Therefore,

$$\begin{aligned} \left| \int_{\partial B(x,\epsilon)} \frac{\partial w}{\partial \nu} v \, dS(z) \right| &\leq C \sup_{\partial B(x,\epsilon)} |v| \int_{\partial B(x,\epsilon)} dS(z) \\ &= C\epsilon^{n-1} \sup_{\partial B(x,\epsilon)} |v| = O(\epsilon) \to 0 \text{ as } \epsilon \to 0. \end{aligned}$$

Next, we look at

$$\int_{\partial B(x,\epsilon)} \frac{\partial v}{\partial \nu} w \, dS(z).$$

Now

$$\int_{\partial B(x,\epsilon)} \frac{\partial v}{\partial \nu} w \, dS(z) = \int_{\partial B(x,\epsilon)} \left[\frac{\partial \Phi}{\partial \nu}(z-x) - \frac{\partial h^x}{\partial \nu}(z) \right] w \, dS(z).$$

First, using the fact that h^x is smooth and w is smooth near x, we see that

$$\begin{aligned}\left| \int_{\partial B(x,\epsilon)} \frac{\partial h^x}{\partial \nu}(z) w \, dS(z) \right| &\leq C \int_{\partial B(x,\epsilon)} dS(z) \\ &\leq C\epsilon^{n-1}.\end{aligned}$$

Therefore,

$$\int_{\partial B(x,\epsilon)} \frac{\partial h^x}{\partial \nu}(z) w \, dS(z) \to 0 \text{ as } \epsilon \to 0.$$

For the other term, we see that

$$\begin{aligned}\int_{\partial B(x,\epsilon)} \frac{\partial \Phi}{\partial \nu}(z-x) w(z) \, dS(z) &= \frac{1}{n\alpha(n)} \int_{\partial B(x,\epsilon)} \frac{1}{|z-x|^{n-1}} w(z) \, dS(z) \\ &= \frac{1}{n\alpha(n)\epsilon^{n-1}} \int_{\partial B(x,\epsilon)} w(z) \, dS(z) \\ &= \fint_{\partial B(x,\epsilon)} w(z) \, dS(z) \to w(x) \text{ as } \epsilon \to 0.\end{aligned}$$

Similarly, the right-hand side converges to $v(y)$. Therefore, the lemma is proven.

□

We will now prove the first of the theorems above. The second one follows similarly.

Proof of Theorem 11.

For u defined by (4.9), the Poisson kernel is given by

$$K(x,y) = \frac{2x_n}{n\alpha(n)} \frac{1}{|x-y|^n} = -\frac{\partial}{\partial y_n} G(x,y).$$

Now, we know that

$$\Delta_y G(x,y) = 0 \text{ for } y \neq x.$$

In addition, using the fact that G is symmetric, we see that

$$\Delta_x G(x,y) = \Delta_x G(y,x) = 0 \text{ for } x \neq y.$$

Therefore,

$$\begin{aligned}\Delta_x u(x) &= -\Delta_x \int_{\partial \mathbb{R}^n_+} \frac{\partial}{\partial y_n} G(x,y) g(y) \, dS(y) \\ &= -\int_{\partial \mathbb{R}^n_+} \Delta_x \nabla_y G(x,y) \cdot \nu(y) g(y) \, dS(y) \\ &= -\int_{\partial \mathbb{R}^n_+} \nabla_y \Delta_x G(x,y) \cdot \nu(y) g(y) \, dS(y) = 0.\end{aligned}$$

Therefore, $\Delta_x u = 0$. Using this same reasoning, and the fact that for each $y \neq x$, $G(x,y) \in C^\infty(\mathbb{R}^n)$ for $x \neq y$, we can show that $u \in C^\infty$.

To show that

$$\lim_{\substack{x \to x_0 \\ x \in \mathbb{R}^n_+}} u(x) = g(x_0) \text{ for all } x_0 \in \partial\mathbb{R}^n_+,$$

we will use the fact that for each $x \in \mathbb{R}^n_+$,

$$\int_{\partial\mathbb{R}^n_+} K(x,y)\, dy = 1. \tag{4.14}$$

This can be seen by a direct calculation, and we omit this proof here.

Using this fact, we proceed as follows. Fix $x_0 \in \partial\mathbb{R}^n_+$, $\epsilon > 0$. We need to show that there exists a $\delta > 0$ such that

$$|u(x) - g(x_0)| < \epsilon$$

for $|x - x_0| < \delta$. Now using (4.14), we see

$$\begin{aligned} |u(x) - g(x_0)| &= \left| \int_{\partial\mathbb{R}^n_+} K(x,y) g(y)\, dy - \int_{\partial\mathbb{R}^n_+} K(x,y) g(x_0)\, dy \right| \\ &= \left| \int_{\partial\mathbb{R}^n_+} K(x,y)[g(y) - g(x_0)]\, dy \right|. \end{aligned}$$

Now, we know that $K(x,y)$ has a singularity at $y = x_0$, and we are considering $K(x,y)$ as $x \to x_0$. Therefore, we divide the integral into two pieces and handle them separately.

$$\begin{aligned} \left| \int_{\partial\mathbb{R}^n_+} K(x,y)[g(y) - g(x_0)]\, dy \right| &= \left| \int_{B(x_0,\gamma)} K(x,y)[g(y) - g(x_0)]\, dy \right| \\ &\quad + \left| \int_{\partial\mathbb{R}^n_+ - B(x_0,\gamma)} K(x,y)[g(y) - g(x_0)]\, dy \right| \\ &\equiv I + J. \end{aligned}$$

We first look at term I. Using the assumption that g is continuous, we have

$$|g(y) - g(x_0)|_{L^\infty(B(x_0,\gamma))} < \frac{\epsilon}{2} \text{ for } \gamma \text{ sufficiently small,}$$

and

$$\int_{B(x_0,\gamma)} K(x,y)\, dy \leq \int_{\mathbb{R}^n_+} K(x,y)\, dy = 1.$$

Therefore, $|I| < \frac{\epsilon}{2}$ for γ sufficiently small.

Now for this choice of γ, consider term J.

$$J = \int_{\partial\mathbb{R}^n_+ \setminus B(x_0,\delta)} K(x,y)[g(y) - g(x_0)]\, dy.$$

On J, $|y - x_0| > \gamma$. Take $|x - x_0| < \gamma/2$. Then

$$|y - x_0| \le |y - x| + |x - x_0| \le |y - x| + \frac{\gamma}{2} \le |y - x| + \frac{1}{2}|y - x_0|.$$

Therefore, $|y - x| \ge \frac{1}{2}|y - x_0|$ for $|x - x_0| < \gamma/2$. This implies $|y - x|^{-n} \le 2^n |y - x_0|^{-n}$. Therefore,

$$\begin{aligned}
|J| &\le 2|g|_{L^\infty} \int_{\partial\mathbb{R}^n_+ - B(x_0,\gamma)} K(x,y)\,dy \\
&= 2|g|_{L^\infty} \int_{\partial\mathbb{R}^n_+ - B(x_0,\gamma)} \frac{x_n}{n\alpha(n)|y - x|^n}\,dy \\
&\le 2|g|_{L^\infty} \int_{\partial\mathbb{R}^n_+ - B(x_0,\gamma)} \frac{2^n x_n}{n\alpha(n)|y - x_0|^n}\,dy \\
&= \frac{2^{n+1} x_n}{n\alpha(n)} |g|_{L^\infty} \int_{\partial\mathbb{R}^n_+ - B(x_0,\gamma)} \frac{1}{|y - x_0|^n}\,dy \to 0 \text{ as } x \to x_0.
\end{aligned}$$

Therefore, we can make J arbitrarily small by choosing x sufficiently close to x_0.

□