We start with the definition of a group, since it involves only one operation.

**Definition 1** A group \((G, \ast)\) is a set \(G\) together with a map \(\ast : G \times G \to G\) with the properties

1. (Associativity) For all \(x, y, z \in G\), \(x \ast (y \ast z) = (x \ast y) \ast z\).
2. (Units) There exists \(e \in G\) such that for all \(x \in G\), \(x \ast e = x = e \ast x\).
3. (Inverses) For all \(x \in G\) there exists \(y \in G\) such that \(x \ast y = e = y \ast x\).

Note that the most conventional notation for a map, such as \(\ast\), is \(\ast(x, y)\); we write however, as usual in this case, \(x \ast y\).

A basic property is that one can talk about the unit, i.e. given (1) and (2), \(e\) is unique:

**Lemma 1** In any group \((G, \ast)\), the unit \(e\) is unique.

*Proof:* Suppose \(e, f \in G\) are units. Then \(e = e \ast f\) since \(f\) is a unit, and \(e \ast f = f\) since \(e\) is a unit. Combining these, \(e = f\). \(\square\)

Note that this proof used only (1) and (2), so it is useful to define a more general notion than that of a group.

**Definition 2** A semigroup \((G, \ast)\) is a set \(G\) together with a map \(\ast : G \times G \to G\) with the properties

1. (Associativity) For all \(x, y, z \in G\), \(x \ast (y \ast z) = (x \ast y) \ast z\).
2. (Units) There exists \(e \in G\) such that for all \(x \in G\), \(x \ast e = x = e \ast x\).

Thus, a semigroup would be a group if each element had an inverse. Notice also that the proof of the above lemma shows that even in a semigroup, the unit is unique.

We also have that inverses are unique in a group. More generally:

**Lemma 2** Suppose that \((G, \ast)\) is a semigroup with unit \(e\), \(x \in G\), and suppose that there exist \(y, z \in G\) such that \(y \ast x = e = x \ast z\). Then \(y = z\).

Notice that if \(G\) is a group, the existence of such a \(y, z\) is guaranteed, even with \(y = z\), by (3). Thus, this lemma says in particular that in a group, inverses are unique.

However, it says more: in a semigroup, any left inverse (if exists) equals any right inverse (if exists). In particular, if both left and right inverses exist, they are both unique: e.g. if \(y, y'\) are left inverses, they are both equal to any left inverse \(z\), and thus to each other.

*Proof:* We have \(y = y \ast e = y \ast (x \ast z)\) where we used that \(e\) is the unit and \(x \ast z = e\). Similarly, \(z = e \ast z = (y \ast x) \ast z\). But by the associativity, \(y \ast (x \ast z) = (y \ast x) \ast z\), so combining these three equations shows that \(z = y\), as desired. \(\square\)

There are many interesting groups, such as \((\mathbb{R}, +)\), \((\mathbb{Z}, +)\), \((\mathbb{Q}, +)\), \((\mathbb{R}^n, +)\), \((\mathbb{R}^+, \cdot)\), where \(\mathbb{R}^+\) consists of the positive reals, as well as semigroups, such as \((\mathbb{R}, \cdot)\) (all non-zero elements have inverses), \((\mathbb{Z}, \cdot)\) (only \(\pm 1\) have inverses). Another group with a different flavor is \((\mathbb{Z}/(n\mathbb{Z}), +)\), the integers modulo \(n \geq 2\) integer: as a set, this can be identified with \(\{0, 1, \ldots, n - 1\}\) (the remainders when dividing by \(n\)), and addition gives the usual sum in \(\mathbb{Z}\), reduced modulo \(n\), so e.g. in \((\mathbb{Z}/(5\mathbb{Z}), +)\), \(2 + 4 = 1\). It is less confusing though to write \(\{[0], \ldots, [n - 1]\}\) for the set, and \([2] + [4] = [1]\) then.

In general, when the operation is understood, one might just write the set for a group or semigroup, i.e. say \(G\) is a group.

Many (semi)groups are commutative; in fact, all of the above examples are:
Definition 3 A commutative, or abelian, semigroup \((G, *)\) is one in which \(x * y = y * x\) for all \(x, y \in G\).

Noncommutative semigroups will play a role in this class, including the set \(M_n\) of \(n \times n\) matrices with matrix multiplication as the operation, which is non-commutative if \(n \geq 2\), and permutations of a finite set \(S\) which is non-commutative if the set has at least 3 elements (this will be discussed when we talk about determinants).

We then can make the following definition:

Definition 4 A field \((F, +, \cdot)\) is a set \(F\) with two maps \(+ : F \times F \to F\) and \(\cdot : F \times F \to F\) such that

1. \((F, +)\) is a commutative group, with unit 0.
2. \((F, \cdot)\) is a commutative semigroup with unit 1 such that \(1 \neq 0\) and such that \(x \neq 0\) implies that \(x\) has a multiplicative inverse (i.e. \(y\) such that \(x \cdot y = 1 = y \cdot x\)).
3. The distributive law holds:

\[ x \cdot (y + z) = x \cdot y + x \cdot z. \]

One usually writes \(-x\) for the additive inverse (inverse with respect to \(+\)), \(x^{-1}\) for the multiplicative inverse.

Examples then include \((\mathbb{R}, +, \cdot)\), \((\mathbb{Q}, +, \cdot)\), and indeed complex numbers \((\mathbb{C}, +, \cdot)\).

A more interesting field is the subset of \(\mathbb{R}\) given by numbers of the form

\[ \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}. \]

The most interesting part in showing that this is a field is that multiplicative inverses exist; that these exist (within this set!) when \(a + b\sqrt{2} \neq 0\) follows from the following computation in \(\mathbb{R}\):

\[ (a + b\sqrt{2})^{-1} = \frac{a - b\sqrt{2}}{a^2 - 2b^2} = (a^2 - 2b^2)^{-1}a - (a^2 - 2b^2)^{-1}b\sqrt{2}. \]

Notice that \((a^2 - 2b^2)^{-1}a, -(a^2 - 2b^2)^{-1}b\) are indeed rational, and \(a^2 - 2b^2 \neq 0\) as follows from Homework 1, problem 4.

Finally, \((\mathbb{Z}/(n\mathbb{Z}), +, \cdot)\) is not a field in general; e.g. if \(n = 6\), \([2] \cdot [3] = [0]\). However, if \(n\) is a prime \(p\), then it is — it is the finite field of \(p = n\) elements.

As an example of a general result in a field:

Lemma 3 If \((F, +, \cdot)\) is a field, then \(0 \cdot x = 0\) for all \(x \in F\).

Proof: Since \(0 = 0 + 0\), we have

\[ 0 \cdot x = (0 + 0) \cdot x = 0 \cdot x + 0 \cdot x, \]

so

\[ 0 = -(0 \cdot x) + (0 \cdot x) = -(0 \cdot x) + (0 \cdot x + 0 \cdot x) = -(0 \cdot x) + 0 \cdot x + 0 \cdot x = 0 + 0 \cdot x = 0 \cdot x, \]

as desired. On the last line, the first equation is that \(-(0 \cdot x)\) is the additive inverse of \(0 \cdot x\), the second substitutes in the previous line, the third is associativity, the fourth is again that \(-(0 \cdot x)\) is the additive inverse of \(0 \cdot x\), while the fifth is that 0 is the additive unit. \(\square\)

Notice that this proof uses the distributive law crucially: this is what links addition (0 is the additive unit!) to multiplication.

For more examples, see Appendix A, Problem 1.1. Note that (ii) is the statement that if \(x, y \neq 0\) then \(x \cdot y \neq 0\), which in particular shows easily that \((\mathbb{Z}/(n\mathbb{Z}), +, \cdot)\) is not a field if \(n \geq 2\) is not a prime.

(There is a bit more work in showing that if \(n = p\) is a prime, this is a field.)