Mathematics Department Stanford University
Math 61CM – Metric spaces

We have talked about the notion of convergence in $\mathbb{R}$:

Definition 1 *A sequence* $\{a_n\}_{n=1}^{\infty}$ *of reals* **converges** *to* $\ell \in \mathbb{R}$ *if for all* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that* $n \in \mathbb{N}$, $n \geq N$ *implies* $|a_n - \ell| < \varepsilon$. *One writes* $\lim a_n = \ell$.

With $\|.\|$ the standard norm in $\mathbb{R}^n$, one makes the analogous definition:

Definition 2 *A sequence* $\{x_n\}_{n=1}^{\infty}$ *of points in* $\mathbb{R}^n$ **converges** *to* $x \in \mathbb{R}^n$ *if for all* $\varepsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that* $n \in \mathbb{N}$, $n \geq N$ *implies* $\|x_n - x\| < \varepsilon$. *One writes* $\lim x_n = x$.

One important consequence of the definition in either case is that limits are unique:

Lemma 1 *Suppose* $\lim x_n = x$ *and* $\lim x_n = y$. *Then* $x = y$.

Proof: Suppose $x \neq y$. Then $\|x - y\| > 0$; let $\varepsilon = \frac{1}{2}\|x - y\|$. Thus there exists N_1 such that $n \geq N_1$ implies $\|x_n - x\| < \varepsilon$, and N_2 such that $n \geq N_2$ implies $\|x_n - y\| < \varepsilon$. Let $n = \max(N_1, N_2)$. Then

$$\|x - y\| \leq \|x - x_n\| + \|x_n - y\| < 2\varepsilon = \|x - y\|,$$

which is a contradiction. Thus, $x = y$. $\square$

Note that the properties of $\|.\|$ were not fully used. What we needed is that the function $d(x, y) = \|x - y\|$ was non-negative, equal to 0 only if $x = y$, symmetric ($d(x, y) = d(y, x)$) and satisfied the triangle inequality. We thus make the following definition.

Definition 3 *A* **metric space** (X, d) *is a set* X *together with a map* $d : X \times X \to \mathbb{R}$ *(called a* **distance function***) such that*

1. $d(x, y) \geq 0$ *for all* $x, y \in X$, *and* $d(x, y) = 0$ *if and only if* $x = y$.
2. $d(x, y) = d(y, x)$ *for all* $x, y \in X$,
3. *(Triangle inequality)* $d(x, z) \leq d(x, y) + d(y, z)$ *for all* $x, y, z \in X$.

Recall that a norm $\|.\| : V \to \mathbb{R}$ on a vector space V over $\mathbb{R}$ (or $\mathbb{C}$) is a map that is

1. positive definite, i.e. $\|x\| \geq 0$ for all $x \in V$ with equality if and only if $x = 0$,
2. absolutely homogeneous, i.e. $\|\lambda x\| = |\lambda| \, \|x\|$ for $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) and $x \in V$,
3. satisfies the triangle inequality, i.e. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in V$.

Then one easily checks that every normed vector space is a metric space with the induced metric $d(x, y) = \|x - y\|$; for instance the triangle inequality for the metric follows from

$$d(x, z) = \|x - z\| = \|(x - y) + (y - z)\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z),$$

where the inequality in the middle is the triangle inequality for norms.

There are many interesting metric spaces that are *not* normed vector spaces. For instance, it is not hard to check that if V is a vector space with a norm, and one defines $d(x, y) = \min(\|x - y\|, 1)$, one gets a metric that is not the metric induced by the norm.

A less interesting, and indeed rather pathological, example is the discrete metric space: let X be any set, and let $d(x, y) = 0$ if $x = y$, $d(x, y) = 1$ if $x \neq y$.

A general construction is the following: if (X, d) is a metric space, and $A \subset X$, then $(A, d|_{A \times A})$ is a metric space, i.e. the metric is $d|_{A \times A}(x, y) = d(x, y)$ for $x, y \in A$. One typically simply writes (A, d) in this case. We call (A, d) the metric space with the **relative metric**.

One can then make the analogous definition of the limit of a sequence in a metric space.

Definition 4 *Suppose $\{x_n\}_{n=1}^\infty$ is a sequence of points in a metric space (X, d) and $x \in X$. One says that $\{x_n\}_{n=1}^\infty$* **converges** *to x, and writes $\lim x_n = x$, if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \in \mathbb{N}$, $n \geq N$ implies that $d(x_n, x) < \varepsilon$.*

With this definition, the same proof as above gives that limits in a metric space are unique.

We now turn to continuity of functions $f : X \to Y$ where (X, d_X), (Y, d_Y) are metric spaces.

Definition 5 *Suppose (X, d_X), (Y, d_Y) are metric spaces. A function $f : X \to Y$ is* **continuous at the point** *$a \in X$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in X$, $d_X(x, a) < \delta$ implies $d_Y(f(x), f(a)) < \varepsilon$.*

A function is called **continuous** *if it is continuous at all $a \in X$.*

Note that this generalizes the usual notion of continuity of real valued functions ($Y = \mathbb{R}$, $d_Y(b, c) = |b - c|$) on subsets X of $\mathbb{R}$ ($d_X(a, x) = |a - x|$): for all $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in X$, $|x - a| < \delta$ implies $|f(x) - f(a)| < \varepsilon$.

There is another closely related notion.

Definition 6 *Suppose (X, d_X), (Y, d_Y) are metric spaces. A function $f : X \to Y$ is* **sequentially continuous at the point** *$a \in X$ if it satisfies that for every sequence $\{x_n\}_{n=1}^\infty$ in X which converges to a, $\lim_{n\to\infty} f(x_n) = f(a)$.*

We then have

Lemma 2 *Suppose (X, d_X), (Y, d_Y) are metric spaces. A function $f : X \to Y$ is continuous at $a \in X$ if and only if it is sequentially continuous at $a \in X$.*

Proof: The direction that sequential continuity implies continuity is Exercise 3.4 in Appendix A of the textbook in the case of real valued functions on intervals; the same proof works in general (and is highly recommended to work this out).

So let us show that continuity at a implies sequential continuity at a. Suppose that f is continuous at a, and that $\{x_n\}_{n=1}^\infty$ is a sequence which converges to a. Let $\varepsilon > 0$. We need to find $N \in \mathbb{N}$ such that $n \geq N$ implies $d_Y(f(x_n), f(a)) < \varepsilon$.

But by the continuity of f at a there exists $\delta > 0$ such that for $x \in X$, $d_X(x, a) < \delta$ implies $d_Y(f(x), f(a)) < \varepsilon$. On the other hand, by the convergence of the sequence of x_n to a, with the definition applied with δ, there exists N such that $n \geq N$ implies $d_X(x_n, a) < \delta$. Thus, for $n \geq N$, $d_X(x_n, a) < \delta$, and so $d_Y(f(x_n), f(a)) < \varepsilon$, which completes the proof as remarked above. $\square$

The Bolzano-Weierstrass theorem on $\mathbb{R}$ stated that bounded sequences had convergent subsequences. In particular, if $[a, b] \subset \mathbb{R}$, and $\{x_n\}_{n=1}^\infty$ is a sequence in $[a, b]$, then it has a subsequence $\{x_{n_k}\}_{k=1}^\infty$ converging to some $x \in [a, b]$. (The limit of any convergent sequence $\{y_n\}_{n=1}^\infty$ in $[a, b]$ is necessarily in $[a, b]$ since $y_n \geq a$ for all n implies $\lim y_n \geq a$, and similarly for b.) The natural generalization is:

Definition 7 *A metric space (X, d_X) is* **(sequentially) compact** *if every sequence $\{x_n\}_{n=1}^\infty$ of points in (X, d_X) has a convergent subsequence.*

With this definition, $[a, b] \subset \mathbb{R}$ is compact.

Here the word ‘sequentially’ refers to the fact that there is in fact an equivalent (in the setting of metric spaces), but more subtle, definition, which is usually called compactness; this other notion in fact generalizes in a useful way beyond even the setting of metric spaces.

We then have:

Theorem 1 *Suppose (X, d) is a compact non-empty metric space, and $f : X \to \mathbb{R}$ is continuous. Then f is bounded, and it attains its maximum and minimum, i.e. there exist points $a, b \in X$ such that $f(a) = \sup\{f(x) : x \in X\}$, $f(b) = \inf\{f(x) : x \in X\}$.*

Proof: Let us show first that $f(X) = \{f(x) : x \in X\}$ is bounded above. Indeed, suppose it is not, i.e. there is no upper bound for $f(X)$. If $n \in \mathbb{N}$, then in particular n is not an upper bound, so there exists $x_n \in X$ such that $f(x_n) > n$. Now consider the sequence $\{x_n\}_{n=1}^{\infty}$. By the compactness of X, it has a convergent subsequence, say $\{x_{n_k}\}_{k=1}^{\infty}$; let's say $x = \lim_{k\to\infty} x_{n_k} \in X$. Due to its continuity, f is sequentially continuous at x, so $\lim_{k\to\infty} f(x_{n_k}) = f(x)$. In particular, applying the definition of convergence with $\varepsilon = 1$, there exists N such that $k \geq N$ implies $|f(x_{n_k}) - f(x)| < 1$. But then

$$|f(x_{n_k})| = |(f(x_{n_k}) - f(x)) + f(x)| \leq |f(x_{n_k}) - f(x)| + |f(x)| < |f(x)| + 1$$

for all $k \geq N$. Since $f(x_{n_k}) > n_k \geq k$ by the very choice of x_{n_k} (here $n_k \geq k$ is true for any subsequence, as is easy to check by induction), this is a contradiction: choose any $k > \max(N, |f(x)| + 1)$, and then the two inequalities are contradictory. This completes the boundedness from above claim; a completely analogous argument shows boundedness from below.

Now, to show that the maximum and minimum are actually attained, consider only the case of the maximum, with that of the minimum being completely analogous. So now $f(X)$ is non-empty (as X is such) and is bounded above, so $M = \sup f(X)$ exists. Then for all $n \in \mathbb{N}$, $M - \frac{1}{n}$ is not an upper bound for $f(X)$, so there exists $x_n \in X$ such that $f(x_n) > M - \frac{1}{n}$. Again, consider the sequence $\{x_n\}_{n=1}^{\infty}$. By the compactness of X, it has a convergent subsequence, say $\{x_{n_k}\}_{k=1}^{\infty}$; let's say $a = \lim_{k\to\infty} x_{n_k} \in X$. We claim that $f(a) = M$. Indeed, due to its continuity, f is sequentially continuous at a, so $\lim_{k\to\infty} f(x_{n_k}) = f(a)$. On the other hand, since $M \geq f(x_{n_k}) > M - \frac{1}{n_k} \geq M - \frac{1}{k}$ by the very choice of the x_n, we have by the sandwich theorem $\lim_{k\to\infty} f(x_{n_k}) = M$. In combination with the just observed sequential continuity, this gives $f(a) = M$, as desired. $\square$

A very similar proof gives:

Theorem 2 *Suppose (X, d_X), (Y, d_Y) are metric spaces, X is compact, and $f : X \to Y$ is continuous. Then $(f(X), d_Y)$ is compact.*

Proof: Suppose that $\{y_n\}_{n=1}^{\infty}$ is a sequence in $f(X)$, i.e. $y_n = f(x_n)$ for some x_n. We need to find a subsequence of $\{y_n\}_{n=1}^{\infty}$ which converges to a point in $f(X)$.

Since X is compact, $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence, say $\{x_{n_k}\}_{k=1}^{\infty}$; let's say $x = \lim_{k\to\infty} x_{n_k} \in X$. We claim that $\{y_{n_k}\}_{k=1}^{\infty}$ converges to $f(x) \in f(X)$. Indeed, by the continuity of f, f is sequentially continuous at x, so $\lim_{k\to\infty} f(x_{n_k}) = f(x)$. As $f(x_{n_k}) = y_{n_k}$, this proves the claim, and thus the theorem. $\square$