Mathematics Department Stanford University
Math 61CM/DM – Multilinear maps

Suppose V is a vector space over a field $\mathbb{F}$ such as $\mathbb{R}$. We have already discussed linear maps from V to another vector space W over $\mathbb{F}$, with their set denoted by $\mathcal{L}(V,W)$. We also discussed that $\mathcal{L}(V,W)$ itself is a vector space over $\mathbb{F}$, with $S+T \in \mathcal{L}(V,W)$ being defined (for $S,T \in \mathcal{L}(V,W)$) by pointwise addition: $(S+T)(x) = Sx + Tx$, $x \in V$, and similarly (for $S \in \mathcal{L}(V,W)$, $c \in \mathbb{F}$) $cS \in \mathcal{L}(V,W)$ being defined by pointwise multiplication: $(cS)(x) = cx$, $x \in V$. (There is also a ring structure via composition when $W = V$, but we don't need this right now.)

A natural question is what is $\dim \mathcal{L}(V,W)$ when V,W are finite dimensional? So let $e_1,\ldots,e_n$ be a basis for V, $e'_1,\ldots,e'_m$ be a basis of W. A linear map $T \in \mathcal{L}(V,W)$ is determined uniquely by its matrix, and this correspondence indeed gives rise to a basis of $\mathcal{L}(V,W)$ under the correspondence of using the basis $E_{k\ell}$, $k=1,\ldots,m$, $\ell=1,\ldots,n$, of $m\times n$ matrices defined by $E_{k\ell}$ having ij entry $=1$ if $i=k$ and $j=\ell$, and $=0$ otherwise. (This corresponds to simply identifying matrices with $\mathbb{F}^{mn}$.) Under the correspondence to linear maps this means that the mn linear maps $L_{k\ell}$, $k=1,\ldots,m$, $\ell=1,\ldots,n$, defined by

$$L_{k\ell}e_j = \begin{cases} e'_k & \text{if } j=\ell \\ 0 & \text{otherwise,} \end{cases}$$

i.e. the linear maps

$$L_{k\ell}(\sum_{j=1}^n x_j e_j) = x_\ell e'_k,$$

give a basis for $\mathcal{L}(V,W)$: if $T \in \mathcal{L}(V,W)$,

$$Te_j = \sum_{i=1}^m t_{ij}e'_i = \sum_{i=1}^m t_{ij}L_{ij}e_j = \sum_{i=1}^m \sum_{\ell=1}^n t_{i\ell}L_{i\ell}e_j$$

for all j, which means that the linear maps T and $\sum_{i=1}^m \sum_{\ell=1}^n t_{i\ell}L_{i\ell}$ are the same (since they agree on a basis, thus on linear combinations of the basis vectors, i.e. on all vectors), so T is a linear combination of the $L_{i\ell}$, so the the latter span $\mathcal{L}(V,W)$. In addition, the $L_{i\ell}$ are linearly independent, for if $\sum_{i=1}^m \sum_{\ell=1}^n t_{i\ell}L_{i\ell} = 0$ then for all j

$$0 = \sum_{i=1}^m \sum_{\ell=1}^n t_{i\ell}L_{i\ell}e_j = \sum_{i=1}^m t_{ij}L_{ij}e_j = \sum_{i=1}^m t_{ij}e'_i,$$

so by the linear independence of the e'_i, $t_{ij}=0$ for all i. This proves that $L_{k\ell}$, $k=1,\ldots,m$, $\ell=1,\ldots,n$, form a basis of $\mathcal{L}(V,W)$, and thus the latter is mn dimensional.

Concretely, in the case of linear maps $\alpha : V \to \mathbb{F}$,

$$\alpha(\sum_{i=1}^n c_i e_i) = \sum_{i=1}^n c_i \alpha(e_i),$$

consider the linear maps f_j specified by $f_j(e_i) = 0$ if $i \neq j$, $f_j(e_i) = 1$ if $i=j$ (commonly written as $f_j(e_i) = \delta_{ij}$ where δ_{ij} is the 'Kronecker delta', i.e. is $=1$ if $i=j$, 0 otherwise), i.e.

$$f_j(\sum_i c_i e_i) = \sum_i c_i f_j(e_i) = c_j.$$

These form a basis for the set V^* of linear maps $V \to \mathbb{R}$ (called the **dual** of V) since any linear map α can be written as above

$$\alpha(\sum_{i=1}^n c_i e_i) = \sum_{i=1}^n c_i\alpha(e_i) = \sum_{i=1}^n (\sum_{j=1}^n f_i(e_j)c_j)\alpha(e_i) = \sum_{i=1}^n \alpha(e_i) f_i(\sum_{j=1}^n c_j e_j),$$

i.e.

$$\alpha = \sum_{i=1}^n \alpha(e_i) f_i.$$

We will need to extend our considerations to multilinear maps. We have already seen an example: inner products were bilinear (linear in each slot) maps $V \times V \to \mathbb{R}$ with certain additional properties. We make the general definition:

Definition 1 *Suppose $V_1, \ldots, V_k, W$ are vector spaces over $\mathbb{F}$. A map $\alpha : \times_{i=1}^k V_i \to W$ is called k-linear if it is linear in each slot, i.e. if for all $j \in \{1, \ldots, k\}$,*

$$\alpha(x_1, \ldots, x_{j-1}, \lambda x_j + \lambda' x_j', x_{j+1}, \ldots, x_k)$$
$$= \lambda\alpha(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_k) + \lambda'\alpha(x_1, \ldots, x_{j-1}, x_j', x_{j+1}, \ldots, x_k)$$

for all $x_i \in V_i$ ($i \in \{1, \ldots k\}$), $x_j' \in V$, $\lambda, \lambda' \in \mathbb{F}$.

So linear maps are 1-linear, bilinear maps are 2-linear, etc.

We are mostly interested in the case when $V_1 = \ldots = V_k = V$, and $W = \mathbb{F}$.

Lemma 1 *The vector space of k-linear maps from V^k to $\mathbb{F}$ is n^k dimensional.*

Proof: Indeed, if $e_1, \ldots, e_n$ is a basis of V, such a map α is determined by its values $\alpha(e_{i_1}, \ldots, e_{i_k})$, $i_1, \ldots, i_k \in \{1, \ldots, n\}$, which values however can be specified freely, for then, for

$$x_1 = \sum_{i_1=1}^n x_{1i_1} e_{i_1}, \ldots, x_k = \sum_{i_k=1}^n x_{ki_k} e_{i_k}, \tag{1}$$

we have

$$\alpha(x_1, \ldots, x_k) = \alpha(\sum_{i_1=1}^n x_{1i_1} e_{i_1}, \ldots, \sum_{i_k=1}^n x_{ki_k} e_{i_k}) = \sum_{i_1=1}^n \ldots \sum_{i_k=1}^n x_{1i_1} \ldots, x_{ki_k} \alpha(e_{i_1}, \ldots, e_{i_k})$$

defines a k-linear map. Correspondingly, a basis is given by the k-linear maps $\alpha_{j_1 \ldots j_k} : \times_{i=1}^k V \to \mathbb{F}$, $j_1, \ldots, j_k \in \{1, \ldots, n\}$ with

$$\alpha_{j_1 \ldots j_k}(e_{i_1}, \ldots, e_{i_k}) = \delta_{i_1 j_1} \ldots \delta_{i_k j_k},$$

as is easy to check similarly to the arguments above. □

Definition 2 *A k-linear map $\alpha : V^k \to \mathbb{F}$ is symmetric if for all $i \neq j$*

$$\alpha(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = \alpha(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n), \qquad x_1, \ldots, x_n \in V,$$

and is called antisymmetric if for all $i \neq j$

$$\alpha(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = -\alpha(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n), \qquad x_1, \ldots, x_n \in V.$$

Thus symmetric maps do not change sign under the interchange of the argument of two slots, while antisymmetric ones do. An example of a symmetric bilinear map is an inner product. For the determinant, however, we will be interested in antisymmetric maps.

Notice also that when V is finite dimensional, for a k-linear map being symmetric is equivalent to the symmetry property when the x_i are basis vectors, for the full result then follows by k-linearity, and similarly for antisymmetry. Indeed, if for all $i : \{1, \ldots, k\} \to \{1, \ldots, n\}$,

$$\alpha(e_{i_1}, \ldots, e_{i_\ell}, \ldots, e_{i_j}, \ldots, e_{i_k}) = \alpha(e_{i_1}, \ldots, e_{i_\ell}, \ldots, e_{i_j}, \ldots, e_{i_k}),$$

then with $x_1, \dots, x_k$ as in (1),

$$\begin{aligned}
&\alpha(x_1, \dots, x_\ell, \dots, x_j, \dots, x_k) \\
&= \alpha(\sum_{i_1=1}^n x_{1i_1} e_{i_1}, \dots, \sum_{i_\ell=1}^n x_{\ell i_\ell} e_{i_\ell}, \dots, \sum_{i_j=1}^n x_{j i_j} e_{i_j}, \dots, \sum_{i_k=1}^n x_{k i_k} e_{i_k}) \\
&= \sum_{i_1=1}^n \dots \sum_{i_k=1}^n x_{1i_1} \dots x_{ki_k} \alpha(e_{i_1}, \dots, e_{i_\ell}, \dots, e_{i_j}, \dots, e_{i_k}) \\
&= \sum_{i_1=1}^n \dots \sum_{i_k=1}^n x_{1i_1} \dots x_{ki_k} \alpha(e_{i_1}, \dots, e_{i_j}, \dots, e_{i_\ell}, \dots, e_{i_k}) \\
&= \alpha(\sum_{i_1=1}^n x_{1i_1} e_{i_1}, \dots, \sum_{i_j=1}^n x_{j i_j} e_{i_j}, \dots, \sum_{i_\ell=1}^n x_{\ell i_\ell} e_{i_\ell}, \dots, \sum_{i_k=1}^n x_{k i_k} e_{i_k}) \\
&= \alpha(x_1, \dots, x_j, \dots, x_\ell, \dots, x_k),
\end{aligned}$$

where on the extreme left and extreme right the argument of the ℓth and jth slots are interchanged.

Both symmetric and anti-symmetric k-linear maps are in particular k-linear maps, and it is easy to see that they form a subspace (sum of two symmetric maps is such, etc.) so they have dimension $\leq n^k$, where $n = \dim V$.

Notice that if α is antisymmetric and k-linear, and $1+1 \neq 0$ in $\mathbb{F}$ (called '$\mathbb{F}$ does not have characteristic 2'), then for $x = x_i = x_j$, $i \neq j$,

$$\begin{aligned}
&\alpha(x_1, \dots, x, \dots, x, \dots, x_n) = \alpha(x_1, \dots, x_i, \dots, x_j, \dots, x_n) \\
&= -\alpha(x_1, \dots, x_j, \dots, x_i, \dots, x_n) = -\alpha(x_1, \dots, x, \dots, x, \dots, x_n),
\end{aligned}$$

so $\alpha(x_1, \dots, x, \dots, x, \dots, x_n) = 0$. In particular,

$$\alpha(e_{i_1}, \dots, e_{i_k}) = 0$$

if two of $i_1, \dots i_k$ are the same, i.e. if $i : \{1, \dots, k\} \to \{1, \dots, n\}$ is not injective.

So, for instance, if $k > n$, there is only one antisymmetric k-linear map: 0. Indeed, there are no injective maps $i : \{1, \dots, k\} \to \{1, \dots, n\}$, so α vanishes identically on any combination of the basis vectors! Thus, the vector space of such maps is 0-dimensional.

Now, if $n = k$, $i : \{1, \dots, k\} \to \{1, \dots, n\}$ not injective is the same as i not bijective, i.e. i not a permutation. Thus, for an antisymmetric n-linear map α

$$\alpha(\sum_{i_1=1}^n x_{1i_1} e_{i_1}, \dots, \sum_{i_n=1}^n x_{ni_n} e_{i_n}) = \sum_{i \in S_n} x_{1i_1} \dots x_{ni_n} \alpha(e_{i_1}, \dots, e_{i_n})$$

Now, if σ is a transposition of m and ℓ, then

$$\alpha(e_{i_1}, \dots, e_{i_n}) = -\alpha(e_{(i\circ\sigma)_1}, \dots, e_{(i\circ\sigma)_n})$$

so by writing i^{-1} as a product of N transpositions, $i^{-1} = \sigma_1 \dots \sigma_N$, we deduce that

$$\alpha(e_{i_1}, \dots, e_{i_n}) = (-1)^N \alpha(e_1, \dots, e_n) = (-1)^{\mathrm{sign}(i)} \alpha(e_1, \dots, e_n).$$

This means that the vector space of n-linear antisymmetric maps on an n-dimensional vector space has dimension ≤ 1 (when the underlying field has characteristic $\neq 2$): namely α is a multiple of the n-linear map specified by

$$\beta(e_{i_1}, \dots, e_{i_n}) = (-1)^{\mathrm{sign}(i)}, \quad i \in S_n,$$

vanishing if i is not a permutation, concretely

$$\alpha = \alpha(e_1, \dots, e_n)\beta,$$

provided such an antisymmetric n-linear map β exists. (If not, the conclusion would be that $\alpha = 0$.)

Now, there is certainly an n-linear map with these properties (one can specify an n-linear map on all combinations of basis vectors arbitrarily, much like discussed above), the question is if it is antisymmetric. But this comes down to the homomorphism property of the sign of permutations. Namely, what we need to check is, with β so defined, if $\beta(e_{i_1}, \dots, e_{i_k}, \dots, e_{i_\ell}, \dots, e_{i_n})$ and $\beta(e_{i_1}, \dots, e_{i_\ell}, \dots, e_{i_k}, \dots, e_{i_n})$ are negatives of each other when $k \neq \ell$. But if σ is the transposition switching k and ℓ, $\beta(e_{i_1}, \dots, e_{i_\ell}, \dots, e_{i_k}, \dots, e_{i_n}) = \beta(e_{(i\circ\sigma)_1}, \dots, e_{(i\circ\sigma)_k}, \dots, e_{(i\circ\sigma)_\ell}, \dots, e_{(i\circ\sigma)_n})$. Thus,

$$\begin{aligned}
&\beta(e_{i_1}, \dots, e_{i_\ell}, \dots, e_{i_k}, \dots, e_{i_n}) = (-1)^{\mathrm{sign}(i\circ\sigma)} = (-1)^{\mathrm{sign}(i)}(-1)^{\mathrm{sign}(\sigma)} \\
&= -(-1)^{\mathrm{sign}(i)} = -\beta(e_{i_1}, \dots, e_{i_k}, \dots, e_{i_\ell}, \dots, e_{i_n}),
\end{aligned}$$

which proves that β is antisymmetric. Thus, the space of n-linear antisymmetric maps on an n-dimensional vector space has dimension 1.

If $V = \mathbb{F}^n$, then we can take the standard basis of $\mathbb{F}^n$, and consider the *unique* antisymmetric n-linear map with

$$\omega(e_1, \dots, e_n) = 1.$$

This is called the determinant.

This is more commonly known for matrices: in that case if A is an $n \times n$ matrix, the columns $Ae_1, \dots, Ae_n$ are n vectors in $\mathbb{F}^n$, and we define

$$\det(A) = \omega(Ae_1, \dots, Ae_n).$$

Expanding:

$$\begin{aligned}
\det(A) &= \omega\Big(\sum_{i_1=1}^{n} a_{i_1 1} e_{i_1}, \dots, \sum_{i_n=1}^{n} a_{i_n n} e_{i_n}\Big) \\
&= \sum_{i_1=1}^{n} \dots \sum_{i_n=1}^{n} a_{i_1 1} \dots a_{i_n n}\,\omega(e_{i_1}, \dots, e_{i_n}) = \sum_{i\in S_n} (-1)^{\mathrm{sign}(i)} a_{i_1 1} \dots a_{i_n n}.
\end{aligned}$$