

Mathematics Department Stanford University
Math 61CM – Open and closed sets

We saw that Bolzano-Weierstrass in \mathbb{R} implies the compactness of intervals $[a, b]$. The analogue of Bolzano-Weierstrass in \mathbb{R}^n also holds:

Theorem 1 Suppose $\{x^{(k)}\}_{k=1}^{\infty}$ is a bounded sequence in \mathbb{R}^n , i.e. that there exists R such that $\|x^{(k)}\| \leq R$ for all k . Then $\{x^{(k_j)}\}_{j=1}^{\infty}$ has a convergent subsequence.

This is Exercise 2.3 in Chapter 2 of the text; it uses that convergence of a sequence $\{x^{(k)}\}_{k=1}^{\infty}$ in \mathbb{R}^n is equivalent to the convergence of the *component sequences* $\{x_j^{(k)}\}_{k=1}^{\infty}$. (See the textbook.)

Recall that the compactness of a metric space (X, d) means that every sequence has a convergent subsequence. If $K \subset X$, equipped with the relative metric, then the statement that (K, d) is compact is equivalent to every sequence in K having a subsequence converging in the space (X, d) to a point $x \in K$. Thus, we can directly talk about a subset of a metric space being compact.

So now suppose that K is a bounded subset of \mathbb{R}^n , and $\{x^{(k)}\}_{k=1}^{\infty}$ is a sequence in K . Then by Bolzano-Weierstrass there is a convergent subsequence $\{x^{(k_j)}\}_{j=1}^{\infty}$ with limit $x = \lim_{j \rightarrow \infty} x^{(k_j)} \in \mathbb{R}^n$. In order to say that K is compact, we need to be able to conclude that $x \in K$ as well. Thus, a very important concept, in a general metric space is:

Definition 1 Suppose C is a subset of a metric space (X, d) . We say that $x \in X$ is a **limit point** of C if there exists a sequence in C converging to x .

A set C in (X, d) is **closed** if it contains all of its limit points.

Note that every point in C is a limit point of C (take the constant sequence x, x, x, \dots); the point of being closed is that the opposite inclusion also holds, i.e. C is exactly the set of its limit points.

Notice also that *closed intervals are closed*, thus their name is reasonable.

Proposition 1 A closed and bounded subset K of \mathbb{R}^n is compact.

Proof: We already know that if K is bounded, every sequence in K has a subsequence converging to some $x \in \mathbb{R}^n$. Thus, x is a limit point of K . Since K is closed, $x \in K$, so indeed every sequence in K has a subsequence converging to a point in K . \square

We discuss the converse direction as well. For this we need the notion of an open ball in a metric space:

$$B_{\rho}(y) = \{x \in X : d(x, y) < \rho\}.$$

A subset A of a non-empty metric space is **bounded** if for some $y \in X$, $\rho > 0$, $A \subset B_{\rho}(y)$. Notice that if $X = \mathbb{R}^n$, this is equivalent to the above notion of boundedness.

In a general metric space, we have:

Proposition 2 If (X, d) is a metric space, and $K \subset X$ is compact then K is closed and bounded.

Proof: Let $y \in X$. Then the distance function from y , $f(x) = d(x, y)$, is continuous on X (check using the triangle inequality), thus by the compactness of K it is bounded on K , so K is bounded (take $\rho > \sup f$, then $K \subset B_{\rho}(y)$).

On the other hand, suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence in K that converges to some $x \in X$. Then every subsequence of $\{x_n\}_{n=1}^{\infty}$ also converges to x . But by compactness of K , there is a subsequence converging to a point in K . Thus, $x \in K$. So K contains all of its limit points, so it is closed. \square

In a general metric space the converse direction fails, i.e. a closed and bounded set is by no means compact. For instance, if one takes a set with the discrete metric, the only convergent sequences are

the eventually (for sufficiently large n) constant ones (take $\varepsilon = 1$ in the definition of convergence), so if one has an infinite set with the discrete metric and takes a sequence that consists of distinct points, it cannot have a convergent subsequence even though in a metric space with the discrete metric every set is closed and bounded.

A more interesting metric space is the following. Suppose (X, d) is a compact metric space, e.g. a compact subset of \mathbb{R}^n . Let $V = C(X)$ be the set of continuous real-valued functions on X . Note that if f is continuous, then so is $|f|$. Thus, in this case, $|f|$ is a bounded function on X (with in fact the maximum attained), so we can let

$$\|f\| = \sup\{|f(x)| : x \in X\}.$$

One checks that this is a norm on V , and concludes that V with the metric

$$d(f, g) = \|f - g\| = \sup|f - g| = \sup\{|f(x) - g(x)| : x \in X\}$$

is a metric space. When $X = [0, 1]$, say, this is an example of a metric space in which closed and bounded sets need not be compact; for instance the closed unit ball

$$B = \{f \in C([0, 1]) : \|f\| \leq 1\}$$

is closed, bounded, but not compact. (An example of a sequence in B without a convergent subsequence is $\{f_n\}_{n=1}^\infty$, $f_n(x) = x^n$ for all n . One checks easily that $\lim f_n(x) = 0$ if $x \neq 0$, $\lim f_n(x) = 1$ if $x = 1$, so if we had a convergent subsequence, it would have to converge to a function f which is 0 on $[0, 1)$, 1 at 1, but there is no such continuous function, i.e. element of V .)

However, we still have the following:

Proposition 3 *Suppose (X, d) a metric space, $K \subset X$ is compact, and $C \subset X$ is closed with $C \subset K$. Then C is compact.*

Thus, in particular (taking $K = X$) closed subsets of compact metric spaces are compact.

Proof: Suppose $\{x_n\}_{n=1}^\infty$ is a sequence in C ; we need to show that it has a subsequence converging to a point in C . But $\{x_n\}_{n=1}^\infty$ is also a sequence in K as $C \subset K$, so by the compactness of K it has a subsequence $\{x_{n_k}\}_{k=1}^\infty$ converging to a point $x \in K$. But $\{x_{n_k}\}_{k=1}^\infty$ is a sequence in C , so this says that x is a limit point of C , so as C is closed, $x \in C$. Thus, $\{x_{n_k}\}_{k=1}^\infty$ gives us a subsequence of $\{x_n\}_{n=1}^\infty$ which converges to a point in C , completing the proof. \square

Here is an application of this, using that on the problem set it was shown that a map is continuous if and only if the inverse image of closed sets is closed.

Theorem 2 *Suppose $f : X \rightarrow Y$ is continuous, bijective, where (X, d_X) and (Y, d_Y) are metric spaces with X compact. Then $f^{-1} : Y \rightarrow X$ is continuous.*

Proof: Note that f being bijective means that the inverse map $f^{-1} : Y \rightarrow X$ exists, it is defined by $f^{-1}(y) = x$ if x is the *unique* point in X such that $f(x) = y$.

In order to minimize confusion, let us denote f^{-1} by g in what follows. By the remark before the statement of the theorem, we only need to show that if $C \subset X$ is closed then $g^{-1}(C) \subset Y$ is closed.

But

$$\begin{aligned} g^{-1}(C) &= \{y \in Y : g(y) \in C\} = \{y \in Y : f^{-1}(y) = x \text{ for some } x \in C\} \\ &= \{y \in Y : y = f(x) \text{ for some } x \in C\} = \{f(x) : x \in C\} = f(C). \end{aligned}$$

Thus, we only need to show that $C \subset X$ closed implied that $f(C)$ is closed in Y . But $C \subset X$ closed and X compact imply that C is compact by the preceding proposition, thus due to the continuity of f ,

$f(C)$ is compact by the theorem at the end of the metric space handout (applied with $f|_C : C \rightarrow Y$), thus $f(C)$ is a closed subset of Y by Proposition 2, completing the proof.

Thus, $g = f^{-1} : Y \rightarrow X$ is continuous. \square

An example of a use of this theorem is to show that the square root function is continuous.

Proposition 4 *The function $g : [0, \infty) \rightarrow [0, \infty)$, $g(x) = \sqrt{x}$, is continuous.*

Note that $f : [0, \infty) \rightarrow [0, \infty)$, $f(x) = x^2$, is bijective by HW#1 Q8. Thus, it has an inverse function g , which is called the square root function. The proposition states that this is continuous.

Proof: The only subtlety is that $[0, \infty)$ is not compact. However, the theorem directly gives that for any $b > 0$, the restriction of g to $[0, b^2]$ (which is the image of $[0, b]$ under f) is continuous since $[0, b]$ is compact. This actually suffices to conclude the continuity of g as follows.

Suppose $a \in [0, \infty)$; we need to show that g is continuous at a . So let $b = \sqrt{a} + 1$. Then by the above remark, g is continuous on $[0, b^2] = [0, a + 1 + 2\sqrt{a}]$. Now let $\varepsilon > 0$. By the continuity of $g|_{[0, b^2]}$ there is $\delta' > 0$ such that $|x - a| < \delta'$, $x \in [0, b^2]$, implies $|g(x) - g(a)| < \varepsilon$. Now let $\delta = \min(1, \delta')$. Then for $x \in [0, \infty)$, $|x - a| < \delta$ implies $|x - a| < 1$, so $x < a + 1 \leq b^2$, so $x \in [0, b^2]$, and moreover $|x - a| < \delta'$, so $|g(x) - g(a)| < \varepsilon$. This completes the proof. \square

Also, recall the following example showing that some assumption like the compactness of the domain is necessary. Let $f : [-2, -1) \cup [0, 1] \rightarrow [-1, 1]$ be given by $f(x) = x$ if $x \in [0, 1]$, $f(x) = x + 1$ if $x \in [-2, -1)$. Then f is continuous, as is easy to check, and it is bijective, with inverse $g : [-1, 1] \rightarrow [-2, -1) \cup [0, 1]$ given by $g(y) = y - 1$ if $y \in [-1, 0)$, $g(y) = y$ if $y \in [0, 1]$. However, g is not continuous at 0: $g(-1/n) = -1 - 1/n$, $g(0) = 0$, and $\lim(-1 - 1/n) = -1 \neq 0$. The issue is, naturally, that $[-2, -1) \cup [0, 1]$ is not compact.

A condition closely related to closedness is openness:

Definition 2 *A subset A of a metric space (X, d) is **open** for for all $x \in A$ there is $\rho > 0$ such that $B_\rho(x) \subset A$.*

Notice that in terms of this notion we have:

$$\lim x_n = x \text{ iff } \forall \rho > 0 \exists N \text{ s.t. } n \geq N \Rightarrow x_n \in B_\rho(x).$$

We then have

Proposition 5 *A subset A of a metric space (X, d) is open if and only if $X \setminus A = A^c$ is closed.*

Proof: Suppose A is open, and let $\{x_n\}_{n=1}^\infty$ be a sequence in A^c converging to some $x \in X$. We need to show that $x \in A^c$. So suppose that on the contrary $x \in A$. By the openness of A there exists $\rho > 0$ such that $B_\rho(x) \subset A$. But $\lim x_n = x$ so there is N such that $n \geq N$ implies $x_n \in B_\rho(x)$, so in particular $x_N \in B_\rho(x) \subset A$, contradicting $x_N \in A^c$. Thus, $x \in A^c$, and so A^c is closed.

Suppose now that A is not open. Then there exists an $x \in A$ such that no ball $B_\rho(x)$, $\rho > 0$, is contained in A . In particular $B_{1/n}(x)$ is not contained in A , i.e. it contains at least one point of A^c . Let $x_n \in B_{1/n}(x) \cap A^c$. Then $\{x_n\}_{n=1}^\infty$ is a sequence in A^c , and $\lim x_n = x \notin A^c$, so A is not closed. \square

The most important properties of open and closed sets are summarized as follows.

Proposition 6 *Suppose (X, d) is a metric space. Then*

1. X, \emptyset are open.
2. If U_1, \dots, U_N are open, then $\bigcap_{n=1}^N U_n$ is open (finite intersection of open sets is open),
3. If $\{U_\alpha : \alpha \in \Gamma\}$ is a collection of open sets, where Γ is a non-empty index set, then $\bigcup_{\alpha \in \Gamma} U_\alpha$ is open (arbitrary union of open sets is open).

Proposition 7 Suppose (X, d) is a metric space. Then

1. X, \emptyset are closed.
2. If C_1, \dots, C_N are closed, then $\bigcup_{n=1}^N C_n$ is closed (finite union of closed sets is closed),
3. If $\{C_\alpha : \alpha \in \Gamma\}$ is a collection of closed sets, where Γ is a non-empty index set, then $\bigcap_{\alpha \in \Gamma} C_\alpha$ is closed (arbitrary intersection of closed sets is closed).

In view of de Morgan's laws and Proposition 5 these two propositions are equivalent. Recall that de Morgan's laws state that

$$(\bigcup_{\alpha \in \Gamma} U_\alpha)^c = \bigcap_{\alpha \in \Gamma} U_\alpha^c$$

and

$$(\bigcap_{\alpha \in \Gamma} U_\alpha)^c = \bigcup_{\alpha \in \Gamma} U_\alpha^c,$$

i.e. the complement of a union is the intersection of complements and vice versa. (See homework.) Moreover, they are all easy to check directly, with, perhaps surprisingly, item (2) being the most sophisticated. For instance, in the open case, if U_1, \dots, U_N are open and $x \in \bigcap_{n=1}^N U_n$, then for all $n = 1, \dots, N$ there exists $\rho_n > 0$ such that $B_{\rho_n}(x) \subset U_n$; then let $\rho = \min\{\rho_1, \dots, \rho_n\}$, and observe that $B_\rho(x) \subset B_{\rho_n}(x) \subset U_n$ for all n , thus $B_\rho(x) \subset \bigcap_{n=1}^N U_n$. (1) is straightforward (for the empty set there is nothing to prove, while for X any ρ , say $\rho = 1$, works), while (3), in the open case, is that if $x \in \bigcup_{\alpha \in \Gamma} U_\alpha$, then $x \in U_\beta$ for some $\beta \in \Gamma$, thus there exists $\rho > 0$ such that $B_\rho(x) \subset U_\beta$, and thus, as $U_\beta \subset \bigcup_{\alpha \in \Gamma} U_\alpha$, $B_\rho(x) \subset \bigcup_{\alpha \in \Gamma} U_\alpha$ as claimed. It is instructive to check the properties of closed sets directly, rather than simply using de Morgan's laws.

One should not think that a 'typical set' in a metric space is either open or closed; typically it is *neither*.

Apart from examples of complements of closed sets, the open ball $B_\rho(y)$, $\rho > 0$, is a good example of an open set. Indeed, if $x \in B_\rho(y)$ then $d(x, y) < \rho$. So let $\delta = \rho - d(x, y) > 0$, and observe that $B_\delta(x) \subset B_\rho(y)$ since if $z \in B_\delta(x)$ then $d(z, y) \leq d(z, x) + d(x, y) < \rho - d(x, y) + d(x, y) = \rho$, where the first inequality is the triangle inequality, and the second one is $d(z, x) < \rho - d(x, y) = \delta$, so $z \in B_\rho(y)$.