Mathematics Department Stanford University
Math 51H – Rearranging series

Recall first that a series $\sum_{n=1}^{\infty} a_n$, where $a_n \in V$, V a normed vector space, converges if the sequence of partial sums, $s_k = \sum_{n=1}^{k} a_n$ does, and one writes

$$\sum_{n=1}^{\infty} a_n = \lim s_k.$$

Recall also that a series converges absolutely if $\sum_{n=1}^{\infty} \|a_n\|$ converges; note that this is a real valued series with non-negative terms. If a_n are real, $\|a_n\|$ is simply $|a_n|$, hence the terminology. We then have:

Theorem 1 *If V is a complete normed vector space, then every absolutely convergent series converges.*

Proof: Suppose $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. Since V is complete, we just need to show that the sequence of partials sums, $\{s_k\}_{k=1}^{\infty}$, $s_k = \sum_{n=1}^{k} a_n$, is Cauchy, since by definition of completeness that implies the convergence of $\{s_k\}_{k=1}^{\infty}$.

But for $n > m$,

$$\|s_n - s_m\| = \|\sum_{j=1}^{n} a_j - \sum_{j=1}^{m} a_j\| = \|\sum_{j=m+1}^{n} a_j\| \le \sum_{j=m+1}^{n} \|a_j\|.$$

The right hand side is exactly the difference between the corresponding partial sums of $\sum_{j=1}^{\infty} \|a_j\|$. Namely, with $\sigma_n = \sum_{j=1}^{n} \|a_j\|$, and for $n > m$, we have

$$|\sigma_n - \sigma_m| = \sigma_n - \sigma_m = \sum_{j=m+1}^{n} \|a_j\|,$$

where we used that $\|a_j\| \ge 0$, so the sequence of partial sums is increasing, in order to drop the absolute value. In combination,

$$\|s_n - s_m\| \le |\sigma_n - \sigma_m|,$$

at first when $n > m$, but the same argument works if $n < m$ with n, m interchanged, and if $n = m$, both sides vanish.

So now to prove that $\{s_k\}_{k=1}^{\infty}$ is Cauchy, let $\varepsilon > 0$. Since $\{\sigma_k\}_{k=1}^{\infty}$ converges, it is Cauchy, so there exists $N \in \mathbb{N}^+$ such that for $n, m \ge N$, $|\sigma_n - \sigma_m| < \varepsilon$. Then for $n, m \ge N$, $\|s_n - s_m\| \le |\sigma_n - \sigma_m| < \varepsilon$, completing the proof. $\square$

While the problem set shows that the rearrangement of series that do not converge absolutely leads to many potential consequences (divergence, convergence to a different limit), absolutely convergent series are well-behaved. First:

Definition 1 *A rearrangement of $\sum_{n=1}^{\infty} a_n$ is a series $\sum_{n=1}^{\infty} a_{j(n)}$, where $j : \mathbb{N}^+ \to \mathbb{N}^+$ is a bijection.*

Let us consider non-negative series first (such as the norms of the terms of an arbitrary series).

Theorem 2 *Suppose $a_n \ge 0$ for all $n \in \mathbb{N}^+$, a_n real. Let S be the set of all finite sums of the a_n, i.e. the set of all sums $\sum_{n \in B} a_n$ where $B \subset \mathbb{N}^+$ is finite. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if S is bounded, and in that case $\sum_{n=1}^{\infty} a_n = \sup S$.*

Proof: Let $s_k = \sum_{n=1}^{k} a_n$ be the kth partial sum, and R be the set of partial sums $\{s_k : k \in \mathbb{N}^+\}$. We already know that the increasing sequence $\{s_k\}_{k=1}^{\infty}$ converges if and only it is bounded above, i.e. iff R is bounded above, and in that case $\lim s_k = \sup R$.

Now $R \subset S$, so if S is bounded above so is R, and $\sup R \leq \sup S$ since $\sup S$ is an upper bound for S, thus for R, and $\sup R$ is the least upper bound.

On the other hand, let $B \subset \mathbb{N}^+$ finite, and let $K = \max B$ (exists because B is finite). Then $s_K = \sum_{n=1}^{K} a_n \geq \sum_{n \in B} a_n$ since $B \subset \{1, 2, \ldots, K\}$ and since $a_n \geq 0$. Thus for all elements $s = \sum_{n \in B} a_n$, B finite, of S, there exists $r = r_K \in R$ such that $r \geq s$. Correspondingly, if R is bounded above, then so is S, with $\sup R \geq r \geq s$ for all $s \in S$, i.e. $\sup R$ is an upper bound for S, so $\sup R \geq \sup S$.

Thus, if either one of S, R is bounded above, so is the other, i.e. both are bounded above, and one has $\sup R \leq \sup S$ as well as $\sup R \geq \sup S$, so the two are equal: $\sup S = \sup R = \sum_{n=1}^{\infty} a_n$. $\square$

As an immediate consequence we have

Theorem 3 *Suppose $a_n \geq 0$ for all n, a_n real, and $\sum_{n=1}^{\infty} a_n$ converges. Then any rearrangement $\sum_{n=1}^{\infty} a_{j(n)}$ converges and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{j(n)}$.*

Proof: This is very easy now: let S be the set of all finite sums of terms in the series as above. By the previous theorem, $\sum_{n=1}^{\infty} a_n$ converges implies that S is bounded above and $\sum_{n=1}^{\infty} a_n = \sup S$. But the set of finite sums of terms of the rearranged series is also S! Thus, again by the previous theorem, the rearranged series also converges, with $\sum_{n=1}^{\infty} a_{j(n)} = \sup S$. Combining these two proves the theorem. $\square$

This can be used to show that real valued absolutely convergent series can be rearranged: write $a_n = p_n - q_n$ with $p_n, q_n \geq 0$ being the 'positive part' and 'negative part' as in the text; if $\sum_{n=1}^{\infty} a_n$ converges absolutely then $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ converge since $p_n, q_n \leq |a_n|$, but these can be rearranged by the previous theorem, to converge to the same limit, and then $\sum_{n=1}^{\infty} a_{j(n)}$ also converges as $a_{j(n)} = p_{j(n)} - q_{j(n)}$, with

$$\sum_{n=1}^{\infty} a_{j(n)} = \sum_{n=1}^{\infty} p_{j(n)} - \sum_{n=1}^{\infty} q_{j(n)} = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n = \sum_{n=1}^{\infty} a_n.$$

The general theorem is

Theorem 4 *If V a complete normed vector space, $\sum_{n=1}^{\infty} a_n$ converges absolutely, then any rearrangement $\sum_{n=1}^{\infty} a_{j(n)}$ converges absolutely and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{j(n)}$.*

Proof: We already know that $\sum_{n=1}^{\infty} a_n$ converging absolutely, i.e. $\sum_{n=1}^{\infty} \|a_n\|$ converging, implies $\sum_{n=1}^{\infty} \|a_{j(n)}\|$ converging, i.e. $\sum_{n=1}^{\infty} a_{j(n)}$ converging absolutely (and in particular converging). Thus, the only remaining statement is to show the equality of the sums: $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{j(n)}$.

The key idea of the proof is that absolute convergence means given any $\varepsilon > 0$ that there are finitely many terms in the series such that if one takes any other finitely many terms, the sum of their norms is $< \varepsilon$.

So let $\varepsilon > 0$. First, since $\sum_{k=1}^{\infty} \|a_k\|$ converges, thus is Cauchy, means that there exists N_1 such that $n, m \geq N_1$ implies $|\sigma_n - \sigma_m| < \varepsilon$, where $\sigma_n = \sum_{i=1}^{n} \|a_i\|$. Thus, for $n > m = N_1$,

$$\sum_{i=N_1+1}^{n} \|a_i\| = \sigma_n - \sigma_{N_1} < \varepsilon.$$

This is exactly the statement that any finitely many of the a_i which do not include $a_1, \ldots, a_{N_1}$ have the sum of their norms $< \varepsilon$. Indeed, suppose $B \subset \mathbb{N}^+$ is finite with all elements $\geq N_1 + 1$. Let $K = \max B$

(finite set, so maximum exists), and observe that for each $i \in B$, $i \in \{N_1+1, N_1+2, \ldots, K\}$. Thus $\sum_{i\in B} \|a_i\| \leq \sum_{i=N_1+1}^{K} \|a_i\| < \varepsilon$. Hence, by the triangle inequality one also has

$$\|\sum_{i\in B} a_i\| \leq \sum_{i\in B} \|a_i\| < \varepsilon.$$

Now, let $s = \sum_{n=1}^{\infty} a_n$, resp. $r = \sum_{n=1}^{\infty} a_{j(n)}$, and let $\{s_k\}_{k=1}^{\infty}$, resp. $\{r_k\}_{k=1}^{\infty}$ be sequence of partial sums of the two series. Let $N_2 = \max A$, $A = \{j^{-1}(1), \ldots, j^{-1}(N_1)\}$, so for $n \geq N_2+1$, $j(n) \notin \{1, \ldots, N_1\}$. Thus, for $n \geq N = \max\{N_1, N_2\}$, the terms of both s_n and r_n include a_i for all $i \leq N_1$. Thus,

$$s_n - r_n = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_{j(i)} = \sum_{i=N_1+1}^{n} a_i - \sum_{i\in\{1,\ldots,n\}\setminus A} a_i,$$

where on the right hand side we dropped $\sum_{i=1}^{N_1} a_i = \sum_{i\in A} a_{j(i)}$ from both sums whose difference we are taking. But $\{N_1+1, \ldots, n\}$ and $\{1, \ldots, n\} \setminus A$ are finite sets disjoint from $\{1, \ldots, N_1\}$. Thus, by the above observation, applied with $B = \{N_1+1, \ldots, n\}$, resp. $B = \{1, \ldots, n\} \setminus A$

$$\Big\| \sum_{i=N_1+1}^{n} a_i \Big\| < \varepsilon, \qquad \Big\| \sum_{i\in\{1,\ldots,n\}\setminus A} a_i \Big\| < \varepsilon.$$

We thus conclude that

$$\|s_n - r_n\| \leq \Big\| \sum_{i=N_1+1}^{n} a_i \Big\| + \Big\| \sum_{i\in\{1,\ldots,n\}\setminus A} a_i \Big\| < 2\varepsilon.$$

In summary, we have shown that for all $\varepsilon > 0$ there exists N such that for $n \geq N$, $|s_n - r_n| < 2\varepsilon$. This shows that $\lim(s_n - r_n) = 0$, and thus $\lim s_n = \lim r_n$, since both sequences of partial sums converge. $\square$