1 Let \( U = (x, u(t)) \), so \( U \) satisfies the initial value problem
\[
U'(t) = (0, F(U(t))) \\
U(0) = (x, y).
\]

Note that the map \( U \mapsto (0, F(U(t))) \) is locally Lipschitz if and only if \( F \) is, and also is \( C^1 \) if and only if \( F \) is. Therefore, the continuity of the map \( (t, x, y) \mapsto \phi_t(x, y) \) follows from the continuity of flows with respect to time and initial conditions, which was proved in class. Moreover, if \( F \) is \( C^1 \), then the fact that the map \( (t, x, y) \mapsto \phi_t(x, y) \) is \( C^1 \) follows from the continuous differentiability offlows of \( C^1 \) vector fields with respect to time and initial conditions, which was also proved in class.

2

a If \( \lambda \) is an eigenvalue of \( A^*A \), then we have a nonzero vector \( x \) so that \( A^*Ax = \lambda x \), so we have
\[
0 \leq |Ax|^2 = x^*A^*Ax = x^*\lambda x = \lambda|x|^2
\]
so \( \lambda \geq 0 \).

b Let \( x_1 \) be such that \( |x_1| = 1 \) and \( |Ax_1| = \max_{|x|=1} |Ax| = \|A\|_{op} \). Thus we must have, for every \( y \perp x_1 \), that
\[
\lim_{\varepsilon \to 0} \frac{|A(x_1 + \varepsilon y)|^2 - |Ax_1|^2}{\varepsilon} = 0,
\]
so
\[
y^*A^*Ax_1 = 0,
\]
which means that \( A^*Ax_1 \) must be a scalar multiple \( \lambda_1 \) of \( x_1 \), so \( \lambda_1 \) is an eigenvalue of \( A^*A \). But then we have that
\[
\|A\|^2_{op} = |Ax_1|^2 = x_1^*A^*Ax_1 = x_1^*\lambda_1 x_1 = \lambda_1 |x_1|^2 = \lambda_1
\]
so the square root of largest eigenvalue of \( A^*A \) is at least \( \|A\|_{op} \). But on the other hand, if \( \lambda \) is an eigenvalue of \( A^*A \) with eigenvector \( x \), normalized so that \( |x| = 1 \), then we have
\[
\|A\|^2_{op} \geq |Ax|^2 = x^*A^*Ax = \lambda |x|^2 = \lambda,
\]
so the square root of the largest eigenvalue of \( A^*A \) is at most \( \|A\|_{op} \).

3

a Since \( \sum_{n=0}^{\infty} a_n z^n \) converges, we must have \( M = \sup_n |a_n z^n| < \infty \). This means that \( |a_n| \leq M |z|^{-n} \) for each \( n \in \mathbb{N} \). Therefore, we have, whenever \( 0 \leq r < |z| \),
\[
\sum_{n=0}^{\infty} |a_n|r^n \leq M \sum_{n=0}^{\infty} \left( \frac{r}{|z|} \right)^n < \infty
\]
since \( r/|z| < 1 \).

b We recall that \( \|A^n\|_{op} \leq \|A\|^n_{op} \). Thus we have that
\[
\sum_{n=0}^{\infty} \|a_n A^n\|_{op} \leq \sum_{n=0}^{\infty} |a_n| \|A\|^n_{op} < \infty
\]
by part (a). Therefore, the sequence \( \sum_{n=0}^{\infty} a_n A^n \) converges absolutely, so it converges since the space of complex \( n \times n \) matrices is a complete metric space.
(a) Notice that, for all \( n \in \mathbb{N}, t, s \in [0,T] \), we have
\[
|x_n(t) - x_n(s)| \leq |t - s| \sup_{r \in [0,T]} |x'(r)| \leq |t - s| \sup_{y \in K} |F(y)|,
\]
and the last supremum is a finite number because \( F \) is continuous and \( K \) is compact. Therefore, the family \( (x_n) \) is uniformly Lipschitz, hence equicontinuous. Moreover, the family \( (x_n) \) is uniformly bounded since \( K \) is a compact subset of \( \mathbb{R}^N \) hence bounded. By the Arzelà-Ascoli theorem, we thus have a subsequence \( n(i) \) and a limit \( x : [0,T] \to K \) so that \( x_{n(i)} \) converges uniformly to \( x \). Therefore, \( x'_{n(i)} = F \circ x_{n(i)} \) converges uniformly to \( F \circ x \). We recall that if \( f_n \to f \) and \( f'_n \to g \) uniformly, then \( f' = g \). This implies that \( x' = F \circ x \).

(b) Suppose that \( x_i \in \overline{B}(p, R/3) \) and that \( x_i \to x \). By part (a), there is a subsequence \( (i_k) \) and a function \( q : [0,\delta] \to \overline{B}(p, R) \) so that \( \phi(\cdot, x_{i_k}) \to q \) as \( k \to \infty \) uniformly on \([0,\delta]\) and that \( q'(t) = F(q(t)) \) for \( t \in [0,\delta] \). The uniform convergence of \( \phi(\cdot, x_{i_k}) \) to \( q \) means in particular that \( q(0) = \lim_{k \to \infty} \phi(0, x_{i_k}) = \lim_{k \to \infty} x_{i_k} = x \). This means that \( q \) is the unique solution to the initial value problem
\[
q'(t) = F(q(t)) \quad q(0) = 0.
\]
This means that if \((t_i, x_i)\) is a sequence in \([0, \delta] \times \overline{B}(p, R/3)\), then for any subsequence \((i_\ell)\), we have a sub-subsequence \((i_{\ell_m})\) so that \( \phi(t_{i_{\ell_m}}, x_{i_{\ell_m}}) \to q(t, x) \). This implies that \( \phi(t_i, x_i) \to q(t, x) \), which is what we wanted to show.

5

\[a\] We have
\[
\frac{d}{dt} |x|^2 = \frac{d}{dt} x' \cdot x = (x')' x + x'x' = x' (A'x + x'Ax) = x' (A' + A)x = 0
\]
since \( A(t) \) is antisymmetric. This means that \( |x|^2 \) is constant, so \( |x| \) is constant.

\[b\] By the above computation, we have that
\[
0 = \left. \frac{d}{dt} |x|^2 \right|_{t=0} = x(0)' (A' + A)x(0).
\]
Since this must be true for every \( x(0) \), this implies that \( A' + A = 0 \), so \( A \) is antisymmetric.

6

We have
\[
(D^2 - 5D + 6)u = 0,
\]
which means that
\[
(D - 3)(D - 2)u = 0.
\]
Let \( w = (D - 2)u \), so \((D - 3)w = 0\). This means that \( Dw = 5w \), so \( w = C_1 e^{3t} \) for some constant \( C_1 \). Therefore, we have \((D - 2)u = C_1 e^{3t} \), so
\[
D(e^{-2t}u) = -2e^{-2t}u + e^{-2t}(C_1 e^{3t} + 2u) = C_1 e^{t},
\]
so
\[
e^{-2t}u = C_1 e^{t} + C_2
\]
for some constant \( C_2 \), so
\[
u = C_1 e^{3t} + C_2 e^{2t}.
\]
Note that
\[
(D^2 - 5D + 6)(C_1 e^{3t} + C_2 e^{2t}) = 9C_1 e^{3t} + 4C_2 e^{2t} - 15C_1 e^{3t} - 10C_2 e^{2t} + 6C_1 e^{3t} + 6C_2 e^{2t} = 0,
\]
so any \( u \) of this form is actually a solution. Therefore, the set of all solutions is
\[
\{ C_1 e^{3t} + C_2 e^{2t} \mid C_1, C_2 \in \mathbb{R} \}.
\]