1. (Differential equations with parameters.) Suppose that $U$ is an open subset of $\mathbb{R}^k$, that $W$ is an open subset of $\mathbb{R}^n$, and that $F : U \times W \to \mathbb{R}^n$ is a locally Lipschitz map. For $x \in U$ and $y \in W$, consider the initial value problem

$$u'(t) = F(x, u(t)), \quad u(0) = y.$$ 

Let $I_{x,y}$ be the largest interval for which a solution exists. Denote the solution by $t \in I_{x,y} \mapsto \phi_t(x,y)$. Let $Q = \{(x,y,t) : x \in U, y \in W, t \in I_{x,y}\}$.

(a). Prove that the map

$$(*) \quad (t,x,y) \in Q \mapsto \phi_t(x,y)$$

is continuous.

(b). If $F$ is $C^1$, show that the map $(*)$ is $C^1$.

[Hint for (a) and (b): there is a way to deduce this (with almost no work) from things we proved in class.]

2. Consider an $n \times n$ complex matrix $A$.

(a). Prove that every eigenvalue of $A^*A$ is real and nonnegative. (Recall that $A^*$ is the matrix whose $ij$ entry is $\bar{a}_{ji}$.)

(b). Show that $\|A\|_{op}$ is equal to the square root of the largest eigenvalue of $A^*A$.

3. Suppose $a_0, a_1, \ldots$ and $z$ are complex numbers such that the series $\sum_{n=0}^{\infty} a_n z^n$ converges.

(a). Prove that if $0 \leq r < |z|$, then $\sum_{n=0}^{\infty} |a_n| r^n < \infty$. [Hint: consider $M = \sup_n |a_n z^n|$.]

(b). Prove that if $A$ is a square, complex matrix with $\|A\|_{op} < |z|$, then $\sum_{n=0}^{\infty} a_n A^n$ converges. (By definition, this means that the sequence $\sum_{n=0}^{m} a_m A^m$ of partial sums converges.)

4. (a). Suppose that $K$ is a compact subset of $\mathbb{R}^N$ and that $F : K \to \mathbb{R}^N$ is a continuous vectorfield. Suppose that $x_n : [0,T] \to K$ is a sequence of functions
such that

\[ x_n'(t) = F(x_n(t)) \quad \text{for all } t \in [0,T]. \]

Prove that \( x_n(t) \) has a subsequence \( x_{n(i)}(t) \) that converges uniformly to a limit \( x : [0,T] \to K \), and that \( x'(t) = F(x(t)) \) for \( t \in [0,T] \).

(b). Suppose that \( F : B(p, R) \subset \mathbb{R}^n \to \mathbb{R}^n \) is a continuous vectorfield and that \( M = \sup |F| < \infty \). Let \( \delta < R/(3M) \). Suppose that for each \( x \in \overline{B(p, R/3)} \), there is a **unique** solution \( u : [0, \delta] \to B(p, R) \) of the initial value problem

\[
\begin{align*}
    u'(t) &= F(u(t)), \\
    u(0) &= x.
\end{align*}
\]

Denote the solution by \( \phi(t,x) \). Show that \((t,x) \in [0,T] \times \overline{B(p, R/3)} \mapsto \phi(t,x)\) is continuous.

[Hint: if suffices to show that if \((t_i,x_i) \in \overline{B(p, R/3)} \times [0,T] \) converges to \((x,t)\), then \( \phi(t_i,x_i) \) converges to \( \phi(t,x) \).]

5. Consider the differential equation:

\[[\star] \quad x'(t) = A'(t)x(t) \]

where \( x : [0,T] \to \mathbb{R}^n \), \( A(t) \) is an \( n \times n \) real matrix, and \( t \mapsto A(t) \) is continuous.

(a). Show that if \( A(t) \) is antisymmetric for each \( t \) and if \( x(\cdot) \) is a solution of \([\star]\), then \( |x(t)| \) is constant.

(b). Show that if \( |x(t)| \) is constant (i.e., independent of \( t \)) for every solution of \([\star]\), then \( A(t) \) is antisymmetric for every \( t \).

6. Let \( D \) be the differentiation operator, i.e., the operator that takes a differentiable function \( t \mapsto u(t) \) to the function \( t \mapsto u'(t) \). Thus \( D^2 \) is the operator that takes the function \( u(\cdot) \) to the function \( u''(\cdot) \) (assuming the second derivative exists). Find all solutions of the differential equation

\[
u'' - 5u' + 6u = 0.
\]

**Hint:** Rewrite the equation as \( D^2u - 5Du + 6u = 0 \), or \( (D^2 - 5D + 6)u = 0 \), or \((D - 3)(D - 2)u = 0\). Let \( w = (D - 2)u \).