(a). We prove the contrapositive. Suppose that \( x_1, \ldots, x_n \) are linearly dependent at some time \( t_1 \). Then there are constants \( c_1, \ldots, c_n \), not all zero, such that
\[
\sum c_i x_i(t_1) = 0.
\]
Then \( x(t) := \sum c_i x_i(t) \) is a solution of the initial value problem
\[
\begin{align*}
x'(t) &= A(t)x(t), \\
x(t_1) &= 0.
\end{align*}
\]
The 0 function is also a solution of this initial value problem. But we know solutions are unique. Thus \( x(\cdot) \equiv 0 \), so \( x_1, \ldots, x_n \) are linearly dependent at all times.

(a). (Alternate solution). Let \( X(t) \) be the \( n \times n \) matrix whose columns are \( x_1(t) \ldots x_n(t) \). Note that
\[
X'(t) = A(t)X(t).
\]
(That is because, by definition of matrix multiplication, column \( j \) of \( A(t)X(t) \) is \( A(t)x_j(t) \).) Since the columns of \( X(t_0) \) are independent, \( \det X(t_0) \neq 0 \). By Liouville’s Theorem (Proposition 3.13 in the text),
\[
\det X(t) = \det X(t_0) e^{\int_{t_0}^t \text{tr} A(s) \, ds}.
\]
Thus \( \det X(t) \neq 0 \) for all \( t \), so the columns of \( X(t) \) are independent for each \( t \).

(b). Trivially, each linear combination of the \( x_i \) is a solution:
\[
\frac{d}{dt} \sum_i c_i x_i(t) = \sum_i c_i x'_i(t) = \sum_i c_i A x_i(t) = A \left( \sum_i c_i x_i(t) \right).
\]
To see that we get all solutions in this way, let \( x(\cdot) \) be any solution of the equation. Since \( x_1(t_0), \ldots, x_n(t_0) \) are \( n \) independent vectors in \( \mathbb{R}^n \), they form a basis for \( \mathbb{R}^n \). Thus \( x(t_0) \) is a linear combination of the \( x_i(t_0) \):
\[
x(t_0) = \sum_i c_i x_i(t_0)
\]
for suitable constants \( c_1, \ldots, c_n \). Now \( x(t) \) and \( \sum_i c_i x_i(t) \) are two solutions of the ODE that are equal at time \( t_0 \). By the uniqueness theorem, they are equal for all \( t \).

2. We have
\[
\frac{dv}{dx} = \frac{1}{x} \frac{dy}{dx} - \frac{y}{x^2} = \frac{\phi(v) - v}{x}.
\]
By separation of variables, we therefore have
\[
\int \frac{dv}{\phi(v) - v} = \int \frac{dx}{x} = \log |x| + C,
\]
so

$$|x| = \tilde{C} \exp \left\{ \int \frac{dv}{\phi(v) - v} \right\}.$$ 

3. This falls under the framework of the previous problem with

$$\phi(v) = \frac{1 + v}{1 - v}.$$ 

Thus we have

$$|x| = \tilde{C} \exp \left\{ \int \frac{dv}{1 + v - v(1 - v)} \right\}$$

$$= \tilde{C} \exp \left\{ \int \frac{1 - v}{1 + v - v(1 - v)} dv \right\}$$

$$= \tilde{C} \exp \left\{ \int \frac{1 - v}{1 + v^2} dv \right\}$$

$$= \tilde{C} \exp \left\{ \arctan v - \frac{1}{2} \log |1 + v^2| \right\}$$

$$= \tilde{C} \exp \left\{ \frac{\arctan(y/x)}{\sqrt{1 + (y/x)^2}} \right\}$$

$$= \tilde{C}|x| \exp \left\{ \frac{\arctan(y/x)}{\sqrt{x^2 + y^2}} \right\}.$$ 

Therefore, we have for $x \neq 0$

$$\tilde{C} \exp \left\{ \frac{\arctan(y/x)}{\sqrt{x^2 + y^2}} \right\} = 1.$$ 

4. (a). The equilibrium points are when $x_2^2 + x_1 x_2 = 2$ and $x_1^2 + x_1 x_2 = 2$. Adding and subtracting the two equations, we see that this occurs when $(x_1 + x_2)^2 = 4$ and $x_1^2 = x_2^2$. The solutions to this system are $(x_1, x_2) = (1, 1)$ and $(x_1, x_2) = (-1, -1)$.

Around $(1, 1)$, the linearized equation is

$$\dot{x}_1' = \ddot{x}_1 + 3\ddot{x}_2$$

$$\dot{x}_2' = 3\ddot{x}_1 + \ddot{x}_2;$$

the eigenvalues of this system are $-2$ and $4$, so this is a hyperbolic equilibrium point and we have a saddle.

Around $(-1, -1)$, the linearized equation is

$$\dot{x}_1' = -\ddot{x}_1 - 3\ddot{x}_2$$

$$\dot{x}_2' = -3\ddot{x}_1 - \ddot{x}_2;$$

the eigenvalues of this system are $2$ and $-4$, so this is a hyperbolic equilibrium point and we have a saddle.

(b). The phase portrait is here:
(c). We have
\begin{align*}
(x_1 + x_2)' &= (x_1 + x_2)^2 - 4 \\
(x_1 - x_2)' &= (x_1 + x_2)(x_1 - x_2)
\end{align*}

From this we see that the curve \( x_1 = x_2 \) is invariant under the evolution. Thus it is the unstable manifold for the equilibrium point \((1, 1)\) and the stable manifold for the equilibrium point \((-1, -1)\). (The stability can be checked by checking the sign of the derivative along the manifold.) We also see that the curves \( x_1 + x_2 = 2 \) and \( x_1 + x_2 = -2 \) are invariant under the evolution. Therefore, the curve \( x_1 + x_2 = 2 \) is the stable manifold for the equilibrium point \((1, 1)\), and \( x_1 + x_2 = -2 \) is the unstable manifold for the equilibrium point \((-1, -1)\).

5. Note that
\[
\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 1 \\ -1 & \lambda - 1 \end{vmatrix} (\lambda - 2) \\
= ((\lambda - 3)(\lambda - 1) + 1)(\lambda - 2) \\
= (\lambda^2 - 4\lambda + 4)(\lambda - 2) \\
= (\lambda - 2)^3.
\]
Consequently the matrix \( N = A - 2I = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \) is nilpotent (indeed \( N^3 = 0 \)) and commutes with \( A \). Thus
\[
e^{At} = e^{t(2I+N)}
= e^{2tI} e^{tN}
= e^{2t} \left( I + tN + \frac{t^2}{2!}N^2 \right)
= e^{2t} \left( I + t \begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \right)
= e^{2t} \begin{bmatrix} (1+t) & -t & -t^2/2 \\ t & (1-t) & t \\ 0 & 0 & 1 \end{bmatrix}
\]

6. Let \( k \) be a positive integer such that \( N^k = 0 \). Then
\[(I + N)(I - N + N^2 - \ldots + N^{k-1}) = I - N^k = I.\]

7. Let \( x \in \mathbb{C}^n \). Then (by (*) in the statement of the problem), there exist \( x_i \in \ker(\lambda_i I - A)^{\nu_i} \) such that
\[x = x_1 + \cdots + x_k.\]
Note that if \( q(z) \) and \( \tilde{q}(z) \) are two polynomials, then \( q(A)\tilde{q}(A) = \tilde{q}(A)q(A) \). Thus
\[
p(A)x_j = \prod_{i=1}^{k} (\lambda_i I - A)^{\nu_i} x_j
= \left( \prod_{i \neq j} (\lambda_i I - A)^{\nu_i} \right) (\lambda_j I - A)^{\nu_j} x_j
= 0.
\]
Thus
\[
p(A)x = p(A) \left( \sum_{j=1}^{k} x_j \right) = \sum_{j=1}^{k} p(A)x_j = 0.
\]
We have shown that \( p(A)x = 0 \) for every vector \( x \). Thus \( p(A) = 0 \).