NORMS ON VECTOR SPACES

Recall that a norm on a real or complex vector space $V$ is a function $F : V \to \mathbb{R}$ such that

\[
F(x + y) \leq F(x) + F(y)
\]

\[
F(cx) = |c|F(x)
\]

\[
F(x) > 0 \quad \text{if} \quad x \neq 0
\]

for all vectors $x$ and $y$ and for all scalars $c$.

We say that two norms $F$ and $G$ on $V$ are equivalent if there is a $\lambda < \infty$ such that

\[
\lambda^{-1}G(x) \leq F(x) \leq \lambda G(x)
\]

for all $x \in V$.

**Theorem 1.** Any two norms on a finite-dimensional vector space are equivalent.

**Proof.** It suffices to prove it in case the vector space is $\mathbb{R}^n$ or $\mathbb{C}^n$ and $G$ is the Euclidean norm $\| \cdot \|$. Let $\mu = \max_{1 \leq i \leq n} F(e_i)$. Then $\mu < \infty$ and

\[
F(x) = F \left( \sum_i x_i e_i \right) \leq \sum_i |x_i| F(e_i) \leq \mu \sum_i |x_i| \leq \mu \sqrt{n} \|x\|
\]

by the Cauchy-Schwartz Inequality (applied to the vectors $x$ and $\sum_i e_i$). Thus

\[
|F(x) - F(y)| = |F(x - y)| \leq \mu \sqrt{n} \|x - y\|
\]

so $F$ is continuous. Hence $F$ restricted to the unit sphere $\{ x : \|x\| = 1 \}$ attains its minimum at some point $p$. Note that $F(p) > 0$. We claim that

\[
F(x) \geq F(p) \|x\|
\]

(1) If $x = 0$, this is trivially true. If $x \neq 0$, then

\[
F(x) = \|x\| F \left( \frac{x}{\|x\|} \right) \geq \|x\| F(p).
\]

By (1) and (2), we are done.\qed

In particular, if $V$ is the space of $n \times n$ complex matrices, then the operator norm $\| \cdot \|_{op}$ is equivalent to any other norm, e.g., to the norm $\left( \sum_{i,j} |a_{ij}|^2 \right)^{1/2}$.

Recall that the operator norm of the $n \times n$ complex matrix $A$ is

\[
\|A\|_{op} = \sup_{x \in \mathbb{C}^n, \|x\| \leq 1} \|Ax\|
\]

\[
= \sup_{x \in \mathbb{C}^n, \|x\| = 1} \|Ax\|
\]

\[
= \sup_{x \in \mathbb{C}^n, x \neq 0} \frac{\|Ax\|}{\|x\|},
\]

where $\| \cdot \|$ denotes the standard Euclidean norm: $\|x\| = \left( \sum_i |x_i|^2 \right)^{1/2}$. 

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