ME111
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Class 8 and 9
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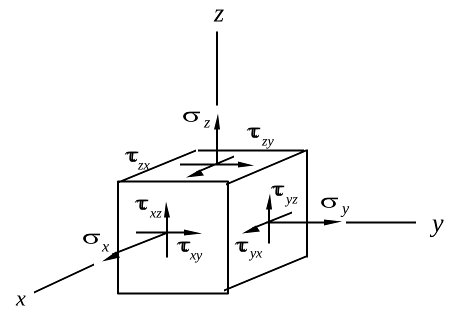
Today's Topics

- Principal stresses in 3-d.
- Maximum shear stresses in plane stress (re-visited).

Reading Assignment Juvinall 4.8 -- 4.11

8.1 States of Stress in 3-d



- The stress tensor now has 9 components which can be expressed in components using matrix notation:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

Only 6 of the components are independent because of symmetry:

$$\tau_{xy} = \tau_{yx}, \quad \tau_{xz} = \tau_{zx}, \quad \tau_{yz} = \tau_{zy}$$

- In 3-d, Mohr's circle is very complicated.

8.2 Principal Stresses in 3-d

- As designers, we wish to know the largest possible stresses that can exist at a point -- these are called the principal stresses.

- These largest stresses (the principle stresses) will usually determine whether failure of the material will occur or not.

- In most materials we will be more concerned about the magnitude of the principal stresses, and less so with their orientations (one exception - anisotropic materials).

Principal Stresses in 3-d (Continued)

- In 3-d there are 3 principal stress values which are the eigenvalues of the following standard eigenvalue problem:

$$\left(\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} - \sigma \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

For a solution to exist, this requires:

$$\det \begin{bmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{bmatrix} = 0$$

- The roots of the determinant polynomial are the principal stresses. The polynomial has the form:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$$

- The quantities I_1, I_2, I_3 are called the stress invariants

$$I_1 = \sigma_x + \sigma_y + \sigma_z$$
$$I_2 = \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x$$
$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2$$

Principal Stresses in 3-d (Continued)

- The principal stresses are ordered so that

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

- The planes on which the principal stresses act are mutually orthogonal.

- Once the principal stresses are known, the magnitude of the maximum shear stresses can be determined from:

$$\tau_{13} = \frac{|\sigma_1 - \sigma_3|}{2}$$
$$\tau_{21} = \frac{|\sigma_2 - \sigma_1|}{2}$$
$$\tau_{32} = \frac{|\sigma_3 - \sigma_2|}{2}$$

Maximum occurring shear stress:

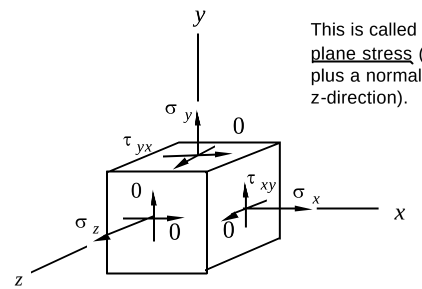
$$\tau_{max} = \max(\tau_{13}, \tau_{21}, \tau_{32})$$

- The maximum shear stresses will occur on planes inclined at 45^0 to the planes of principal stresses, and will also be mutually orthogonal.

8.3 Triaxial Stress States

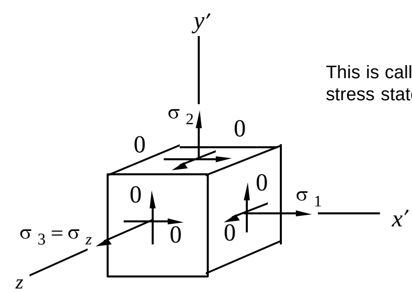
- There is one special case of 3-d stresses that commonly arises that can can be easily analyzed.

- Consider the special case where $\tau_{zx} = \tau_{xz} = 0$



This is called generalized plane stress (plane stress plus a normal stress in the z-direction).

- Transforming the stresses in the x-y plane to principal axes gives:



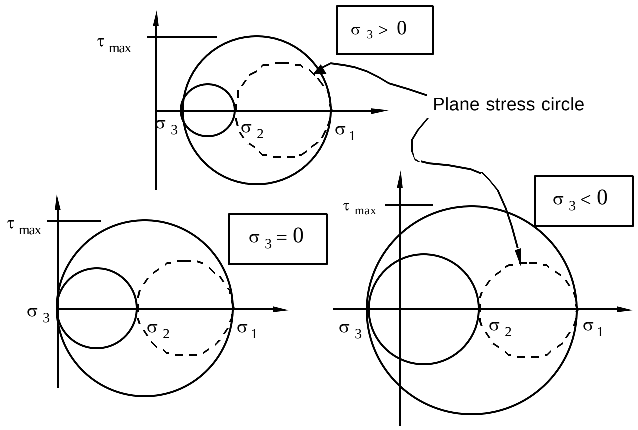
This is called a triaxial stress state.

- We know that the maximum shear stresses will occur on planes at 45^0 to the principal planes.
- The maximum shear stresses are:

$$\tau_{13} = \frac{|\sigma_1 - \sigma_3|}{2} \qquad \tau_{21} = \frac{|\sigma_2 - \sigma_1|}{2} \qquad \tau_{32} = \frac{|\sigma_3 - \sigma_2|}{2}$$

$$\tau_{max} = \max(\tau_{13}, \tau_{21}, \tau_{32})$$

- Graphically, using Mohr's circle, we have (assuming, for example, that $\sigma_1 > \sigma_2 > 0$ and $\sigma_1 > \sigma_2 > \sigma_3$):

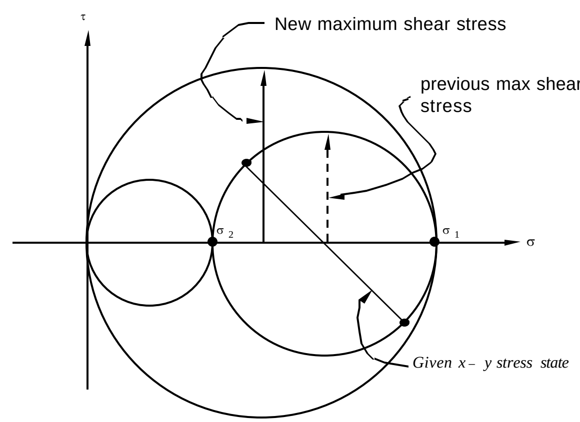


8.4 Maximum Shear Stresses in Plane Strain (Re-Visited).

- Once we have the two principal stresses in a plane stress problem, we recognize that the third principal stress (in the z-direction) is zero.
- The maximum shear stress occurring is then:

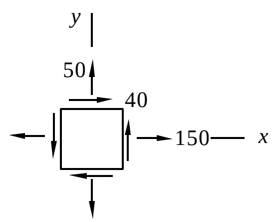
$$\tau_{max} = \max(\tau_{13}, \tau_{21}, \tau_{32}) \quad \text{where} \quad \tau_{13} = \frac{|\sigma_1 - 0|}{2}, \quad \tau_{21} = \frac{|\sigma_2 - \sigma_1|}{2}, \quad \tau_{32} = \frac{|0 - \sigma_2|}{2}$$

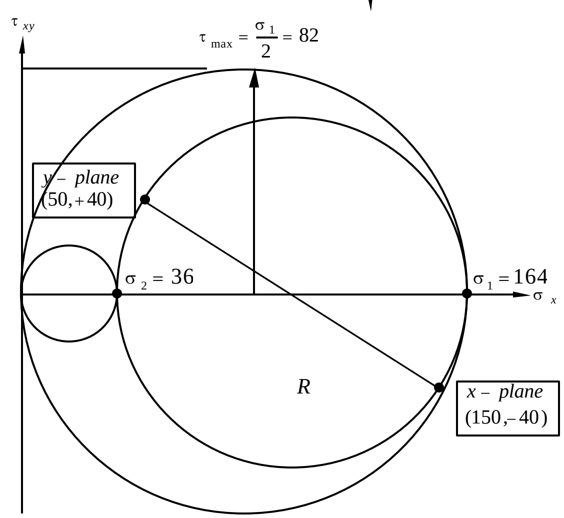
Example:



Example 7.4 (Re-Visited)

Given $\sigma_x = 150MPa,$
$\sigma_y = 50MPa,$
$\tau_{xy} = 40MPa$





Example 7.5 (Re-Visited)

Given $\sigma_x = -50MPa,$
$\sigma_y = 10MPa,$
$\tau_{xy} = -40\,MPa$