August 8, 2009

Stability of rotation about principle moments of inertia

Part 1:

Give each student a book, it should be hard bound, not too many pages and not too heavy. An example might be Griffith's quantum mechanics book. A tennis racquet is also an excellent way to describe the effect they should seek.

- Have the students tape the book shut (or place a rubber band around it) so that it does not open.
- Ask the students to explore the nature of the book as a rotating rigid body. Have them note any observations they have in their notebooks.

* The goal is to get them to observe the instability in rotations about the intermediate principle axis. Give them about 5 or 10 mins or so and then seed them some further thoughts. (Some, especially Eric, may recall me discussing this effect before).

a) *If they have not discovered the effect.* Ask them to note any symmetries of the book and to estimate (calculate if we have a scale) the moment of inertias related to these symmetries. If it takes too long at this point. Ask them to analyze rotations about these symmetry axes.

b) *If they have discovered the effect.* Have them carefully articulate clearly to you (and note in lab book) what the effect is. Guide them to state which axis is different. If necessary, give them other items (cell phone, tennis racquet) to explore to refine their conclusion.

Then ask them what might be causing this. Let them stew over it for another 15 minutes, writing possible causes for this effect.

- After they have written possible causes or elements related to it. Have them tell you what their thoughts are. You can ask them questions to narrow, or eliminate, the possibilities.

  If they say 'air-resistance', you can ask "Do you think this would occur in outer space?"

  If they say Earth's gravity is a cause, ask "Do you think this would occur in a weightless environment?". At this point you can show them a NASA video with someone performing this same experiment (turn sound off as he may explain it). I have to dig up the link to this.

- If time is getting short, ask them what basic laws and principles of physics are at play. You could seed them by listing out laws and principles and have them include or ignore them.
- They should reach a list like: Newton's laws of motion, conservation of energy, conservation of momentum, conservation of angular momentum, and perhaps others that might be explained away. (Of course it is only energy and angular momentum that we care about). They might eliminate momentum conservation by having the books move with different momenta to see if it changes the result. (NASA video I think shows the item in rotation but fixed in space). They can explore varying the rotation rate as well (e.g. if you don't rotate it do you see the effect? no, duh).

  In the end they should carefully articulate what the effect is, what they propose might be the cause, and what physics principles are at play.

Part II

Now just have them think about the principles involved and see how far they get in mathematically representing the system and employing the principles. You may need to prompt them to write down the conservation laws. (I assume at this point they have boiled it down to energy & angular momentum.

You may need to have them write down the rotational KE for each axis in addition to the angular momentum relations as well. Guide them to express the two equations in comparable forms (e.g. express the energy equation in terms of L_i^2).

Solution

The conservation of energy and angular momentum relations can both be expressed in terms of L^2 and then compared.

$$\begin{aligned} E &= \frac{1}{2}I_{xx}\omega_x^2 + \frac{1}{2}I_{yy}\omega_x^2 + \frac{1}{2}I_{zz}\omega_x^2 \\ \vec{L} &= \vec{L}_x + \vec{L}_y + \vec{L}_z \end{aligned}$$

Since we can not compare a scalar, E, and a vector, $\vec{L}$ (use old joke about mountain climber and mosquito) we would like to express $\vec{L}$ as a scalar. Since it is conserved, its square is also conserved.

$$L^2 = \vec{L} \cdot \vec{L} \quad = \quad L_x^2 + L_y^2 + L_z^2$$

where each angular momentum term can be represented as $L_x^2 = I_{xx}^2\omega_x^2$ etc. We can relate these two by either writing them in common terms of ω_i of L_i. It turns out to be easier to write them in terms of the latter.

$$\begin{aligned} E &= \frac{L_x^2}{2I_{xx}} + \frac{L_y^2}{2I_{yy}} + \frac{L_z^2}{2I_{zz}} \\ L^2 &= L_x^2 + L_y^2 + L_z^2 \end{aligned}$$

Both of these expressions must be satisfied for the rotating body. The terms E, L^2, I_{ii} are all constants and L_i^2 is the variable. Notice that the angular momentum constraint is the equation for a sphere in the space defined by axes L_x^2, L_y^2, L_z^2. and the energy constraint surface is an ellipsoid if the three principle moments of inertia are unequal. Since both of these equations must be satisfied, allowed states of rotation will consist of the *intersection* of these two surfaces. We can tie this more directly to the vector $\vec{L}$ by examining these constraints in the space of angular momentum: clearly, since L^2, being thought of as the squared length of the vector, is a constant, the end of the $\vec{L}$ vector lies on a sphere of radius L. The energy surface results in an ellipsoid in the space of $\vec{L}$.

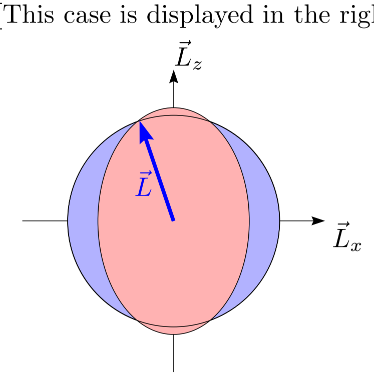
To analyze the situation at hand, let's take $I_{xx} < I_{yy} < I_{zz}$ and examine rotations close to each of these axes. (These can be envisioned by thinking of a basketball and a football where the football can vary in size).

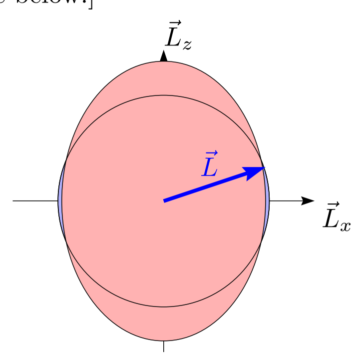
Rotations *close* to largest principle moment of inertia, I_{zz}

In this case the tip of the angular momentum vector is slightly displaced from the L_z axis. The angular momentum sphere is of radius L and the energy ellipsoid *must* intersect the sphere at the tip (or else the two constraints are not being satisfied. Notice that if I_{zz} is the largest of the three principle moments and from the expression above, $E \sim \frac{L_z^2}{2I_{zz}}$, that the ellipsoid s mostly within the sphere and only penetrates near the L_z axis. The intersection for these two surfaces forms a circle that surrounds the L_z axis. Thus, the angular momentum vector, if initiated on this curve, will remain on this curve. I.e. $\vec{L}$ remains close to the z axis, resulting in a stable oscillation about this axis. [See the left figure below.]

Rotations *close* to smallest principle moment of inertia, I_{xx}

Using the same analysis in this case we see that if $_{xx}$ is the smallest then $E \sim \frac{L_z^2}{2I_{xx}}$ which results in the ellipsoid primarily being larger than the sphere except at the intersection near the L_x axis. In this case we see that the intersection line surrounds the L_x axis as it did in the previous case. The result is again a stable rotation for slight deviations about the x axis. [This case is displayed in the right figure below.]





Rotations *close* to intermediate principle moment of inertia, I_{yy}

Applying the same analysis we note the intersection must be near the L_y axis. However, note now that the intersection lines do not surround the L_y axis but form lines that traverse all the way to the other side. I.e. the angular momentum vector will travel along this line, obtaining significant values of L_x and L_z. I.e. the rotations about the intermediate principle axis *is unstable.*