# ME469A

Numerical Methods for Fluid Mechanics

Handout #4

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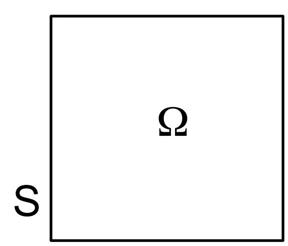
## Finite Volume Methods

Starting point are the conservation law in integral form

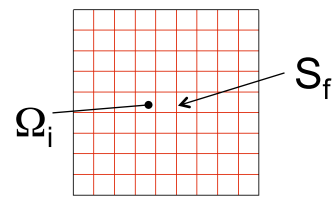



$$\frac{d}{dt}\int_{\Omega} \rho\phi dV + \int_{S} \rho\phi\vec{v}\cdot\hat{n}dS = \int_{S} \Gamma\nabla\phi\cdot\hat{n}dS + \int_{\Omega} Q_{\phi}dV$$

Need to define a discretization of the domain of interest and then transform the continuous equations in a discrete (algebraic) set of equations

# FV Domain Discretization

Divide Ω in subdomains and apply the same governing equation



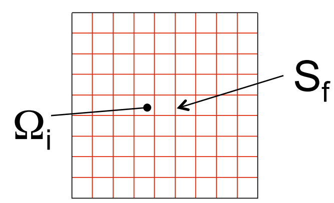

$$\frac{d}{dt}\int_{\Omega_i}\rho\phi dV + \sum_f \int_{S_f}\rho\phi\vec{v}\cdot\hat{n_f}dS = \sum_f \int_{S_f}\Gamma\nabla\phi\cdot\hat{n_f}dS + \int_{\Omega_i}Q_\phi dV$$

In a FD the grid defines the location of the unknowns, in a FV it defines the boundary of the control volumes

# FV Time Term

The rate of change

$$\frac{d}{dt}\int_{\Omega_i}\rho\phi dV$$

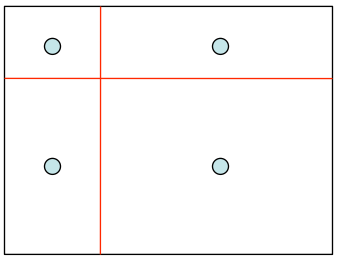


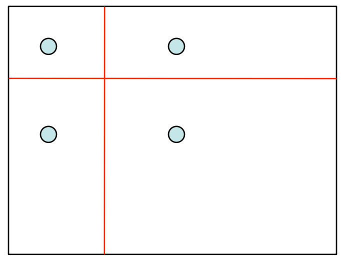

Naturally refers to the evolution of a spatial average of $\rho\phi$ within each elementary volume

Once we define the (discrete) volume integration rules, the original equation becomes an ODE

# FV Domain Discretization

The unknowns are connected to volumes



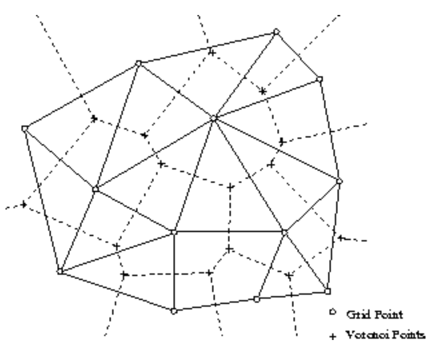


Nodes➔Volumes➔Centers   Centers➔Volumes➔Nodes

Primal Grid   Dual Grid

# FV Domain Discretization

The difference between the two constructions is more clear if we consider irregular (unstructured) grids




Primal Grid
Solid line (triangles)

Dual Grid
Dotted line (polygons)

# FV Surface Integrals

The computation of the fluxes requires the determination of surface integrals, for example the convective fluxes are:

$$\sum_f \int_{S_f} \rho \phi \vec{v} \cdot \hat{n_f} dS$$

Need to use two levels of approximation:

1. the integral is approximated in terms of values defined at one or more points on the cell faces
2. the face values are approximated in terms of the cell center values

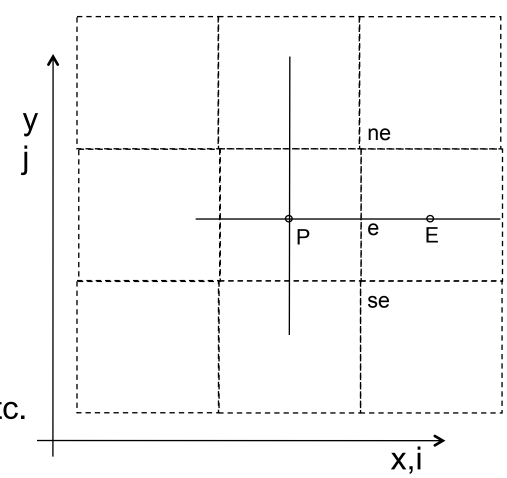
# Nomenclature

Refer to a 2D Cartesian (structured, equally spaced) grid.

It is common to refer to the cells as (i,j) or (P)

The neighbors are (i+1,j)-(i-1,j), etc or (E), (W), etc.

Faces and nodes are lower case (e), (w), etc.



# Surface integrals

Refer to the “east” face of cell P

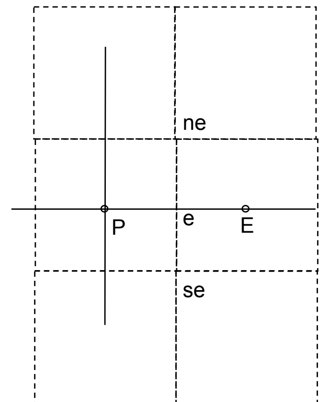
Step 1: Computing the integral

Midpoint Rule:

$$F_e = \int_{S_e} f dS \approx f_e S_e$$

Trapezoidal Rule:

$$F_e = \int_{S_e} f dS \approx \frac{S_e}{2}\left(f_{ne} + f_{se}\right)$$



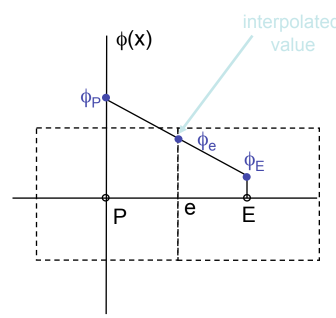

Both second order if the integrand
is evaluated with second order accuracy

# Interpolation Practices

Step 2: compute the function values at the face

Simplest and most intuitive: linear interpolation

$$\phi_e = \phi_E \lambda_e + \phi_P (1 - \lambda_e)$$




where

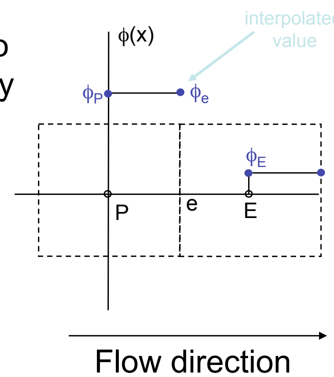
$$\lambda_e = \frac{x_e - x_P}{x_E - x_P}$$

Second order accurate – CDS (central)

# Interpolation Practices

An alternative widely used approach is to consider the direction of propagation of the information:

The convective flux is associated to transport by a “background” velocity field




$$\phi_e = \phi_P \quad \text{if} \quad (\vec{v}\cdot\hat{n})_e > 0$$

$$\phi_e = \phi_E \quad \text{if} \quad (\vec{v}\cdot\hat{n})_e < 0$$

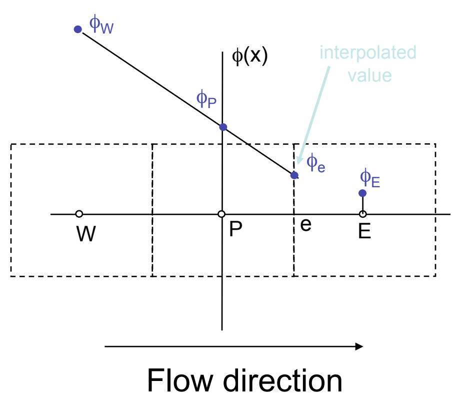
First order accurate – UDS (Upwind)

# Interpolation Practices

The “going-with-the-wind” idea can be extended to high order easily…

$$\phi_e = \phi_P\lambda_w + \phi_W\lambda_e$$

$$\text{if} \quad (\vec{v}\cdot\hat{n})_e > 0$$




Etc.

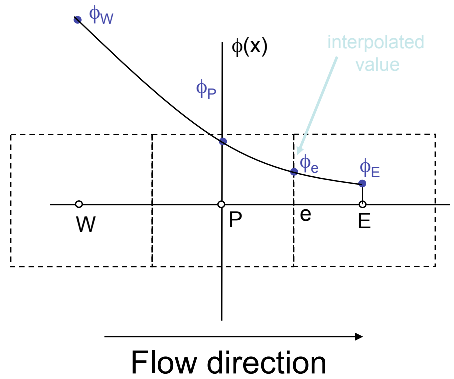
Second order accurate – Note: it is an extrapolation!

# Interpolation Practices

Another common used interpolation extends the idea of upwinding further by considering a parabola *(QUICK: quadratic upwind interpolation for convective kinematics)*

$$\phi_e = g_1\phi_E - g_2\phi_W + $$
$$+ (1 - g_1 + g_2)\phi_P$$
$$\text{if} \quad (\vec{v}\cdot\hat{n})_e > 0$$

$g_i$ are the geometric factors




On a Cartesian uniform grid:

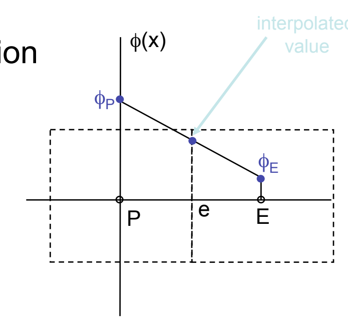
$$\phi_e = \frac{6}{8}\phi_P + \frac{3}{8}\phi_E - \frac{1}{8}\phi_W$$

# Diffusive Fluxes

Diffusive fluxes are NOT associated to an underlying velocity (no basis for upwinding) and involve gradients (Fick's law)

The linear interpolation scheme naturally provides an approximation for the gradient

$$\left(\frac{\partial\phi}{\partial x}\right)_e \approx \frac{\phi_E - \phi_P}{x_E - x_P}$$




What is the accuracy?

# Diffusive Fluxes

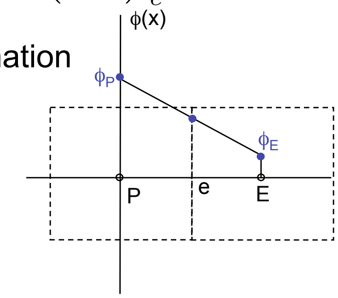
By using Taylor series expansions around $\phi_e$ we can derive

$$\epsilon_\tau = \frac{(x_e - x_P)^2 - (x_E - x_e)^2}{2(x_E - x_P)} \left(\frac{\partial^2 \phi}{\partial x^2}\right)_e + \frac{(x_e - x_P)^3 + (x_E - x_e)^3}{6(x_E - x_P)} \left(\frac{\partial^3 \phi}{\partial x^3}\right)_e + \text{H.O.T.}$$

we obtain a second order approximation

1. If the grid is uniform
2. If the face center is at

$$x_e = \frac{x_E + x_P}{2}$$




# Volume integrals

Similar logic as the surface integrals

Second order:

$$Q_P = \int_{\Omega_P} q dV \approx q_P \Delta V$$

Fourth order (on Cartesian uniform grids):

$$Q_P = \int_{\Omega_P} q dV \approx \frac{\Delta V}{16} (16 q_P + 4 q_s + 4 q_n + 4 q_w + 4 q_e + q_{se} + q_{sw} + q_{ne} + q_{nw})$$

# FV Domain Discretization

Primal Grid – Cell Based Discretization: the unknowns are stored in the volume baricenter ➔ good approximation of the volume integrals

Dual Grid – Vertex/Node Bades Discretization: the faces are mid-way between the cell centers ➔ good approximation of the surface integrals

A third alternative is to use a staggered arrangement, in which the unknowns are not co-located

# Discretized Equations

Recall the original equation

$$\frac{d}{dt}\int_{\Omega_i}\rho\phi dV + \sum_f \int_{S_f}\rho\phi\vec{v}\cdot\hat{n_f}dS = \sum_f \int_{S_f}\Gamma\nabla\phi\cdot\hat{n_f}dS + \int_{\Omega_i}Q_\phi dV$$

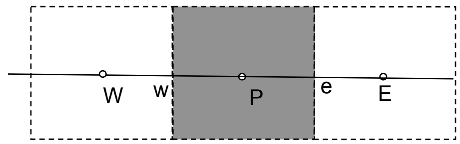
Consider a 1D, steady problem *(d/dt=0),* a conservative variable *($Q_\phi$=0)* and ρ, v and $\Gamma$ constant

$$\sum_f \int_{S_f}\rho\phi\vec{v}\cdot\hat{n_f}dS = \sum_f \int_{S_f}\Gamma\nabla\phi\cdot\hat{n_f}dS$$

It is not of practical relevance because it is not likely that diffusion and convection in 1D problems, but it is interesting from the numerical point of view

# Discretized Equations

1D grid



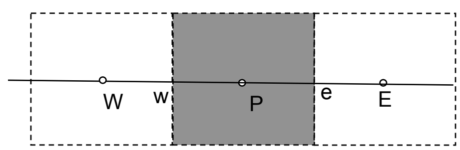

Diffusive terms: linear interpolation

$$\Gamma \nabla \phi \cdot \hat{n}_e = \left( \Gamma \frac{\partial \phi}{\partial x} \right)_e \approx \Gamma \frac{\phi_E - \phi_P}{x_E - x_P}$$

$$\Gamma \nabla \phi \cdot \hat{n}_w = \left( \Gamma \frac{\partial \phi}{\partial x} \right)_w \approx \Gamma \frac{\phi_P - \phi_W}{x_P - x_W}$$

# Discretized Equations

1D grid



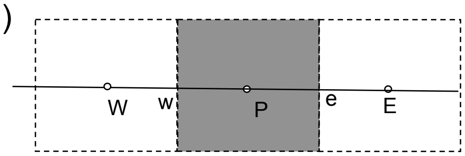

Convective terms: linear interpolation

$$\rho \phi \vec{v} \cdot \hat{n}_e = \rho v \phi_e \approx \phi_E \lambda_e + \phi_P (1 - \lambda_e)$$

$$\rho \phi \vec{v} \cdot \hat{n}_w = \rho v \phi_w \approx \phi_P \lambda_w + \phi_W (1 - \lambda_w)$$

# Discretized Equations

Integrating using the midpoint rule, the face area cancels out (it's 1D – no need to integrate)




Assembling the various terms

$$A_P \phi_P + A_E \phi_E + A_W \phi_W = 0$$

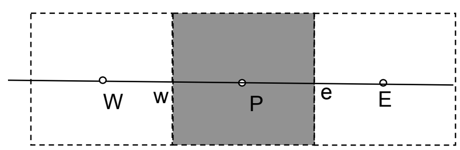
where $A_E = \lambda_e - \dfrac{\Gamma}{x_E - x_P}$ Etc.

With the interesting property: $A_P = -(A_E + A_W)$

# Discretized Equations

Resulting discrete problem is a tridiagonal linear system

$A_P \phi_P + A_E \phi_E + A_W \phi_W = 0$

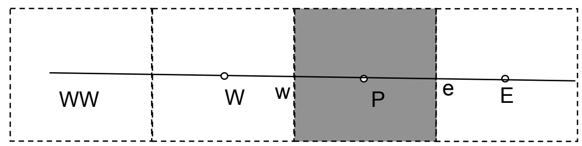



$$\begin{pmatrix} \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & \\ & A_W & A_P & A_E & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \end{pmatrix}$$

Each row corresponds to a cell The sparsity patters corresponds to the cell connectivity (cell-to-cell)

# Discretized Equations

If the discretization was based on the QUICK scheme (assuming v>0) we would obtain




$$A_{WW}\phi_{WW} + A_W\phi_W + A_P\phi_P + A_E\phi_E = 0$$

$$\begin{pmatrix} \cdot & \cdot & & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & \\ & A_{WW} & A_W & A_P & A_E & \\ & & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \end{pmatrix}$$

# Deferred Correction

High order discretization methods naturally introduce larger stencils (more neighbors) more complex filling patterns and reduced sparsity

One of the consequence is that it is NOT possible to use simple “direct” solvers, and typically iterative linear system solver are employed (for general matrices)

This lead to the introduction of a mixed scheme to compute the fluxes

$$F_e = F_C^L + (F_e^H - F_e^L)^{old}$$

The high order term is computed using “old” values within the iterative scheme (right hand side)

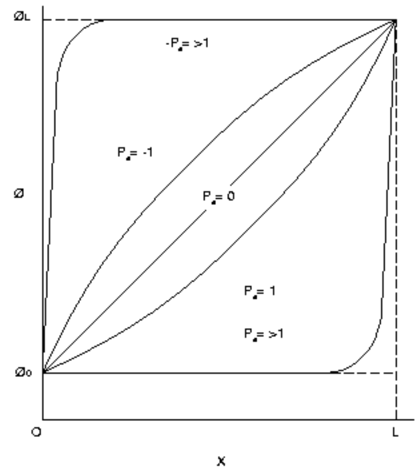
# Exact Solution

This (atypical) problem has an exact solution is Dirichlet conditions are applied to x=0 ($\phi_0$) and x=L ($\phi_L$)

$$\frac{\phi(x)-\phi_0}{\phi_L-\phi_0}=\frac{\exp(Pe\frac{x}{L})-1}{\exp(Pe)-1}$$

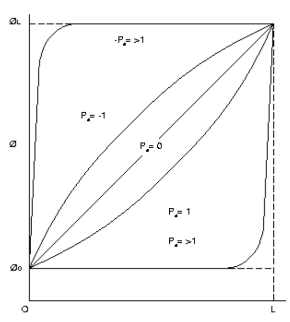
The Peclet number is

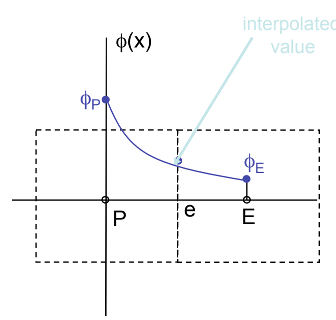
$$Pe=\frac{\rho u L}{\Gamma}$$




# Power-Law Scheme

The exact solution can be used in an unusual way to develop an interpolation scheme






$$\frac{\phi(x)-\phi_0}{\phi_L-\phi_0}=\frac{\exp(Pe\frac{x}{L})-1}{\exp(Pe)-1} \qquad \frac{\phi_e-\phi_P}{\phi_E-\phi_P}=\frac{\exp\left[Pex_e/(x_E-x_P)\right]-1}{\text{exp(Pe)-1}}$$