ME469A
Handout #7

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Iterative Solvers for Linear Systems

A generic iterative method can be written as:

$$M\phi^{p+1} = N\phi^{p} + B$$

- At convergence $\phi^{p+1} = \phi^{p} = \phi$ which implies $A = M - N, \quad Q = B$
- The convergence increases if $M \approx A \rightarrow N \approx 0$
- We can use $M = diag(A)$ or $M = lower(A)$ which lead to systems that are easy to solve.
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Line-by-line solvers

Consider the 5-point discretization we introduced to solve the steady convection/diffusion problem in 2D

$$A_S\phi_S + A_W\phi_W + A_P\phi_P + A_N\phi_N + A_E\phi_E = Q_P$$

It is possible to define an iterative system as:

$$A_W\phi_W^{p+1} + A_P\phi_P^{p+1} + A_E\phi_E^{p+1} = Q_P - A_S\phi_S^{p} - A_N\phi_N^{p}$$

- Each $W - E$ line is treated implicitly, the $S - N$ lines are explicit! It is obviously possible to treat the $S - N$ line implicitly.

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Incomplete LU (ILU) Decomposition

- We mentioned that the *LU* decomposition leads to an excellent solver, although it requires high computational cost and does not take advantage of the matrix sparsity.

  Idea: We can use an approximate factorization of the matrix

  Assume $M = LU$ where L and U are constructed using the Gauss elimination steps but remain sparse!
- We simply ignore (i.e. set to zero) all the entries that are originally zero in A - *incomplete Cholesky Factorization* - not very effective
- We allow for an increase in the matrix filling - *Stone's methods, strong implicit procedure*

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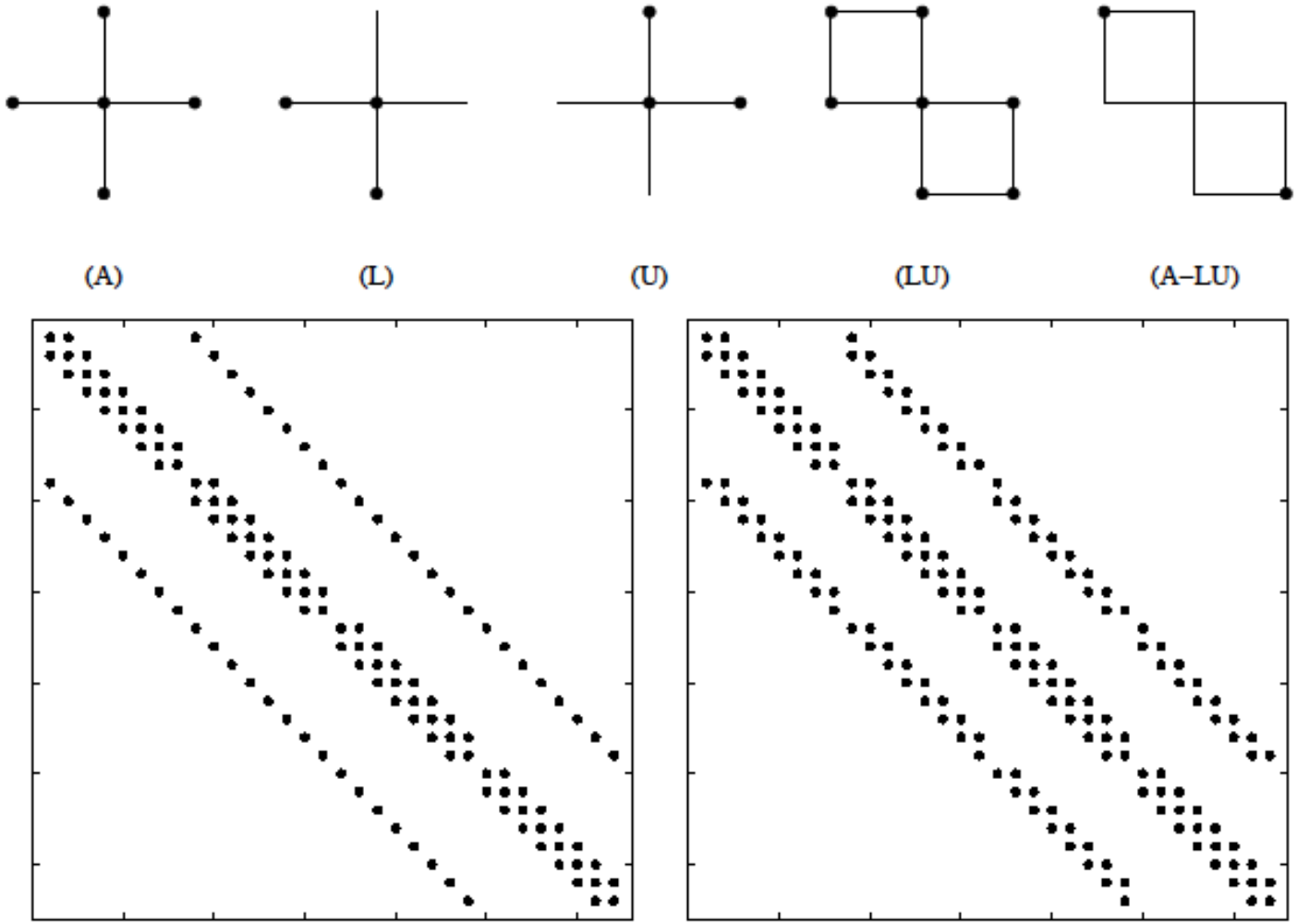
LU decomposition

Consider a 5-point stencil (2D convection/diffusion equation):



Stone's Method

- Consider the effect of the matrix M on the vector ϕ:
  $(M\phi)_P = M_P\phi_P + M_S\phi_S + M_N\phi_N + M_E\phi_E + M_W\phi_W +$
  $M_{SE}\phi_{SE} + M_{NW}\phi_{NW}$
- N should contain the additional diagonals because
  $N = M - A = ILU - A$
- We also want $N\phi \approx 0$ which results in
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- The elements of N should be such that
  $N_P\phi_P + \sum_{nb} N_{nb}\phi_{nb} \approx M_{SE}\phi_{SE} + M_{NW}\phi_{NW}$

Remark: $\sum_{nb} N_{nb}\phi_{nb}$ is a short hand notation for $N_S\phi_S + N_N\phi_N + N_E\phi_E + N_W\phi_W$

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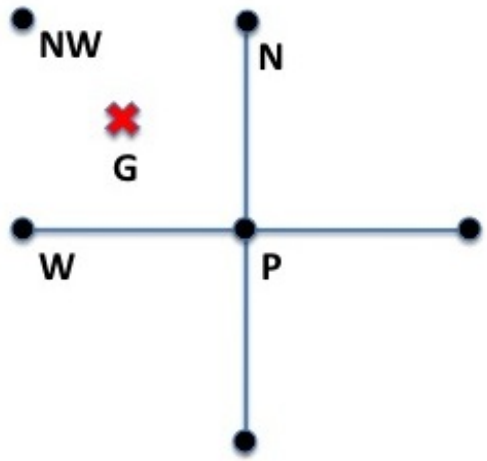
- Stone showed that is possible to obtain a unique splitting (M and N) by choosing

  $N_P\phi_P + \sum_{nb} N_{nb}\phi_{nb} \approx M_{SE}\phi^*_{SE} + M_{NW}\phi^*_{NW}$

  where ϕ^* are interpolated values

- For example $\phi^*_{NW} \approx \alpha(\phi_W + \phi_N - \phi_P)$ is obtained equating $(\phi_W + \phi_N)/2$ and $(\phi^*_{NW} + \phi_P)/2$ which are two approximations of ϕ_G
- Note: $\alpha < 1$ for stability
- This is NOT a general algorithm



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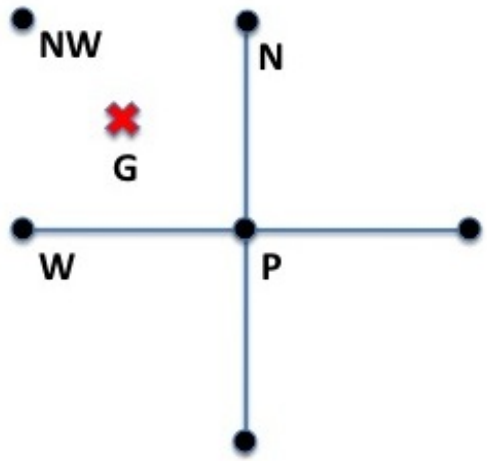

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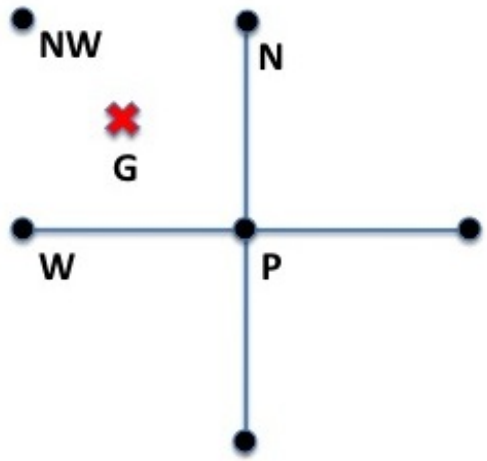

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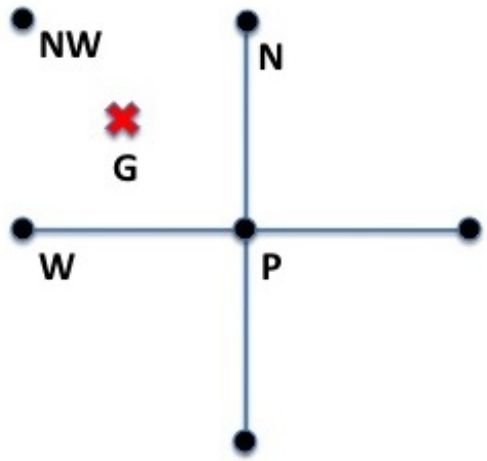

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ILU(T) methods

The idea behind the Stone's method can be generalized to allow for an increase in filling for the *LU* factors.

- Cholesky's and Stone's methods are examples of fill/drop by position
- ILU(T) methods are typically based on fill/drop by *numerical* size: T is the size threshold.

Extensive literature, for example see Y. Saad

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Splitting Methods

- Originally developed for the time-dependent heat condition equation

$$\frac{\partial \phi}{\partial t} = \Gamma \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right)$$

- Consider a FD discretization scheme (Crank-Nicholson)

$$\frac{\phi_i^{n+1} - \phi_i^n}{\Delta t} = \frac{\Gamma}{2} \left[\left(\frac{\delta^2 \phi}{\delta x^2} \right)_i^{n+1} + \left(\frac{\delta^2 \phi}{\delta y^2} \right)_i^{n+1} + \left(\frac{\delta^2 \phi}{\delta x^2} \right)_i^{n} + \left(\frac{\delta^2 \phi}{\delta y^2} \right)_i^{n} \right]$$

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- It is possible to manipulate the discretized equations to obtain

$$\left(1-\frac{\Gamma\Delta t}{2}\frac{\delta^2}{\delta x^2}\right)\left(1-\frac{\Gamma\Delta t}{2}\frac{\delta^2}{\delta y^2}\right)\phi^{n+1}=$$
$$\left(1+\frac{\Gamma\Delta t}{2}\frac{\delta^2}{\delta x^2}\right)\left(1+\frac{\Gamma\Delta t}{2}\frac{\delta^2}{\delta y^2}\right)\phi^{n}-\frac{(\Gamma\Delta t)^2}{4}\frac{\delta^2}{\delta x^2}\left[\frac{\delta^2(\phi^{n+1}-\phi^n)}{\delta y^2}\right]$$

- The last contribution can be rewritten noting that:

$$\phi^{n+1}-\phi^n\approx\Delta t\frac{\partial\phi}{\partial t}$$

- The last term is $O(\Delta t)^3$ and therefore can be ignored! (the approximation of the time derivative is second order)

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- This is equivalent to a two-step solution:

$$\left(1-\frac{\Gamma\Delta t}{2}\frac{\delta^2}{\delta x^2}\right)\phi^{*}=\left(1+\frac{\Gamma\Delta t}{2}\frac{\delta^2}{\delta y^2}\right)\phi^{n}$$
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Generalized Splitting Methods

- The splitting of the operators used in the ADI is based on the fact that the discrete operators ($\delta^2/\delta x^2$ and $\delta^2/\delta y^2$) are decoupled!
- This decoupling is typical of structured grids but does not apply in general
- It is possible to generate the splitting in a general fashion:

  $A = L + U$ (additive decomposition, as opposed to multiplicative $A = LU$).
- The solution procedure is then:

  $(I - L\Delta t)\phi^* = (I + U\Delta t)\phi^n$

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- The splitting of the operators used in the ADI is based on the fact that the discrete operators ($\delta^2/\delta x^2$ and $\delta^2/\delta y^2$) are decoupled!
- This decoupling is typical of structured grids but does not apply in general
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- Many of the methods summarized before use directly the ordering of the unknowns and therefore are limited to structured grids
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Geometrical Example

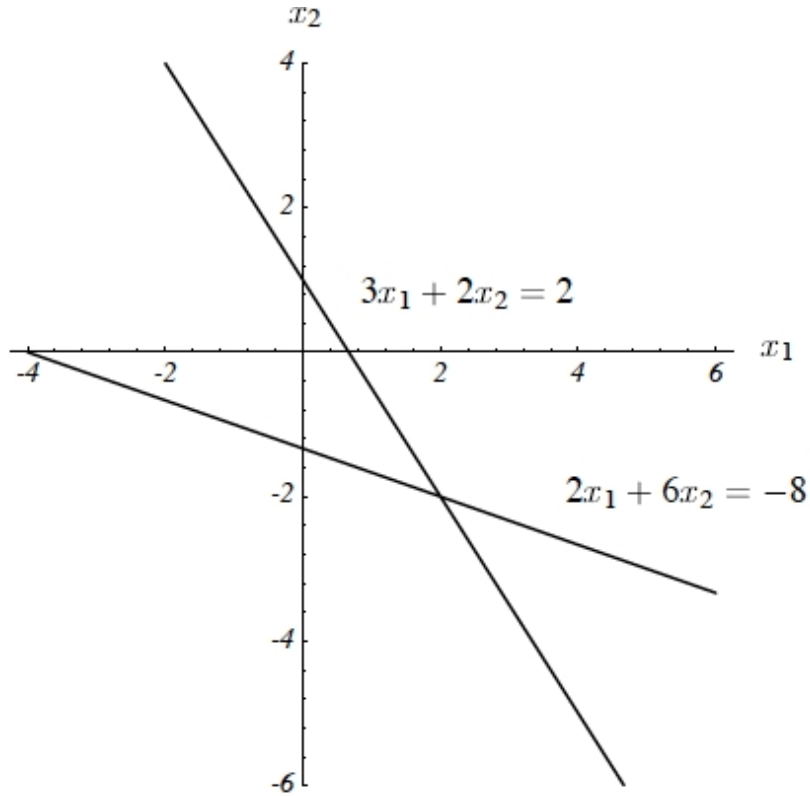
Consider a two-equation system

$$3x_1 + 2x_2 = 2$$
$$2x_1 + 6x_2 = -8$$

$$Ax = b$$

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}, b = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$$

The solution is $x = [2, -2]^T$



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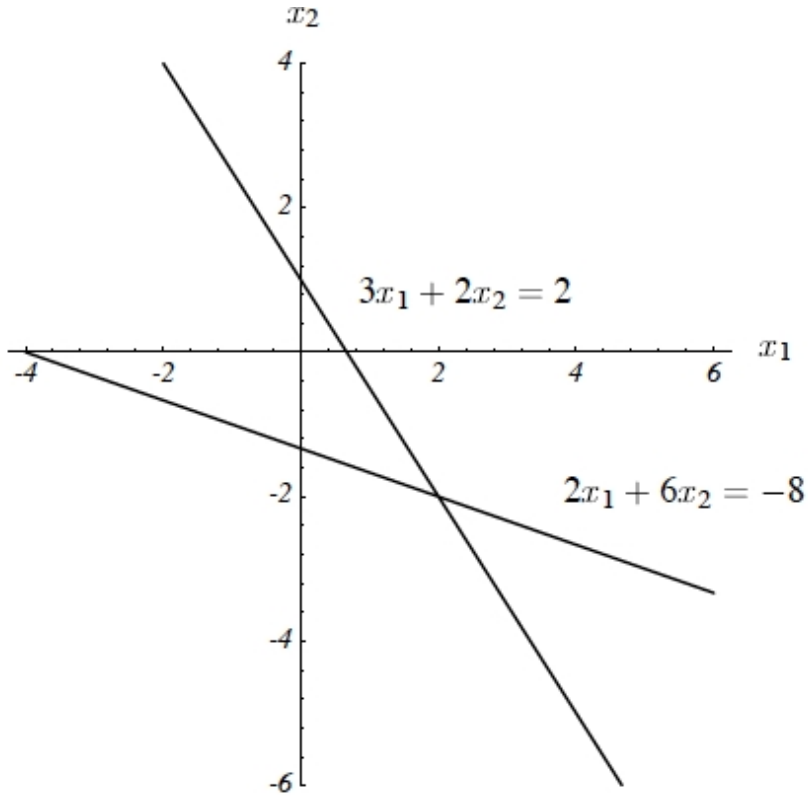

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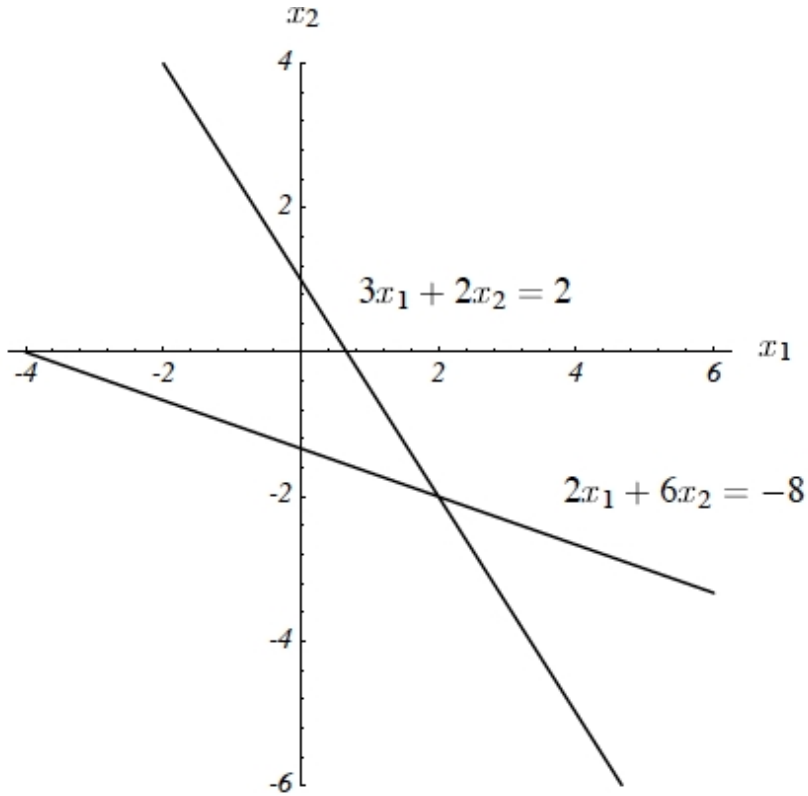

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- Consider an iterative method: $Mx^{p+1} = Nx^p + b$ with $A = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}, b = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$
- Jacobi's iteration is based on $M = Diag(A) = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}, N = M - A = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$
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Recall that the error evolution is $\epsilon^{p+1} = (M^{-1}N)\epsilon^p$

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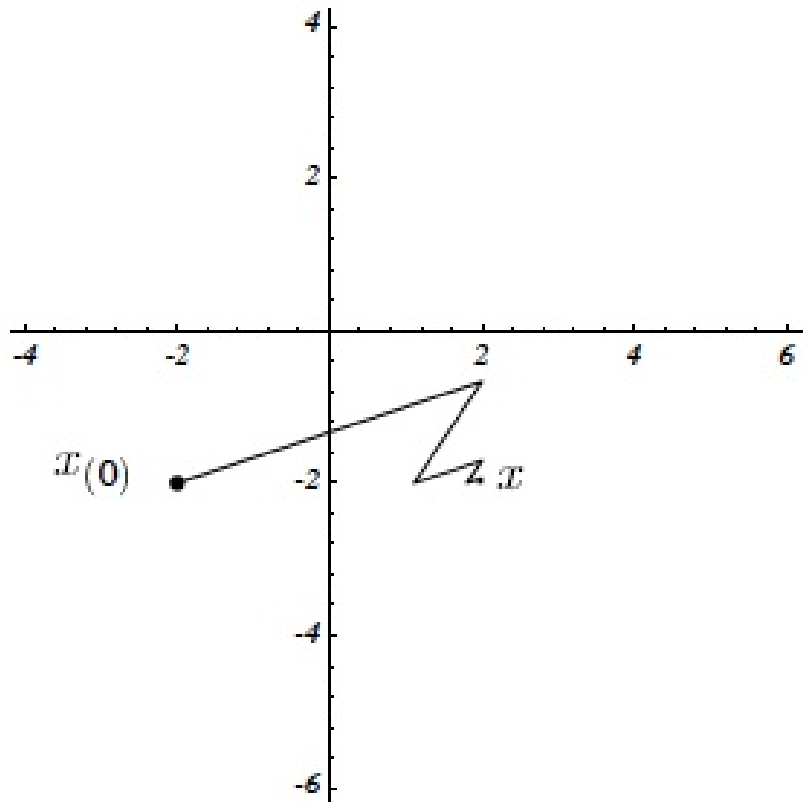

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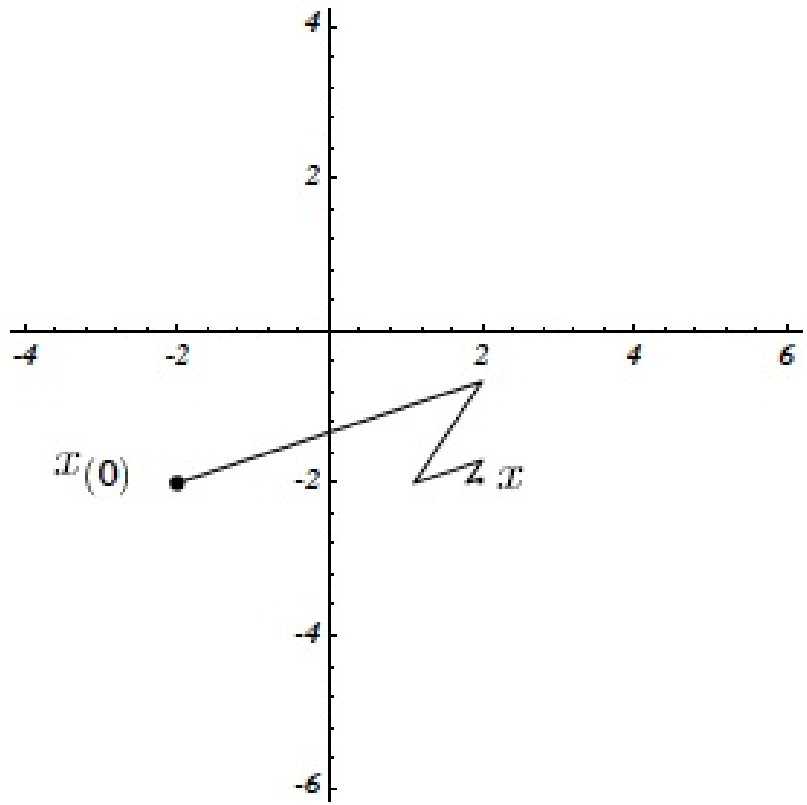

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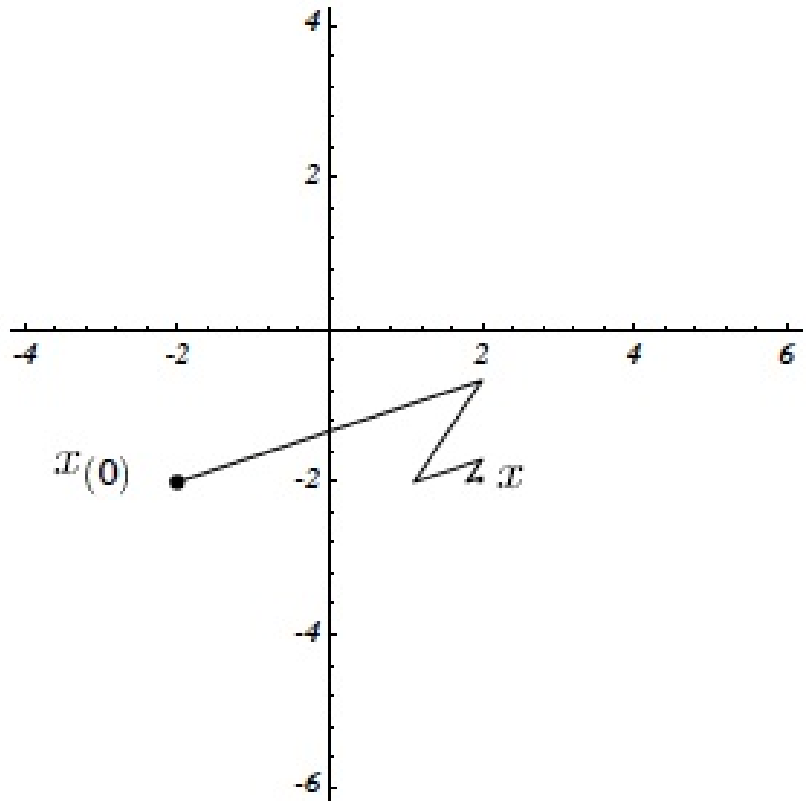

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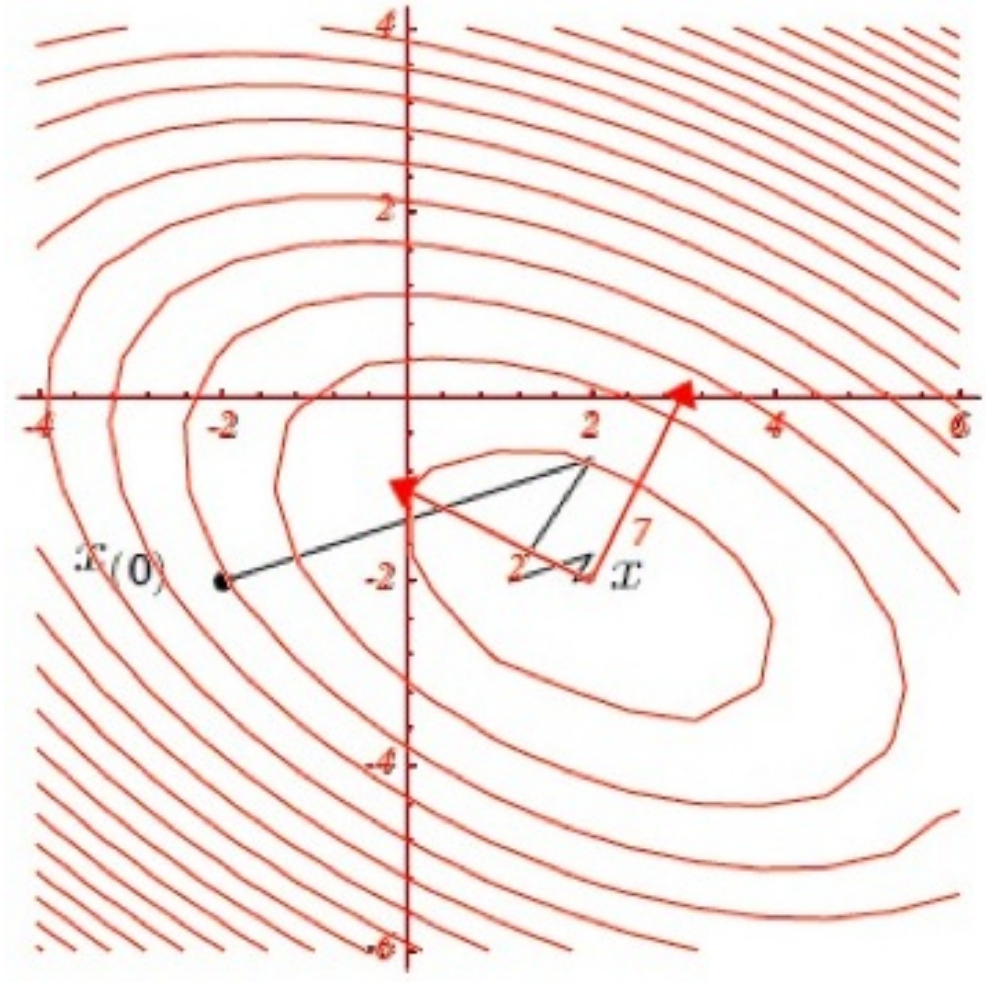

Build a terrain that we can navigate towards the solution

Minimization framework

- Given $Ax = b$ consider the quadratic form $f(x) = x^T Ax/2 - b^T x + c$
- The gradient of $f(x)$ is $f'(x) = A^T x/2 + Ax/2 - b$
- If A is symmetric then $f'(x) = Ax - b$
- if A is positive definite, than $f(x)$ is a paraboloid and $f'(x) = 0$ is a minimum!
- Instead of solving $Ax = b$ we can minimize the quadratic form $f(x)$
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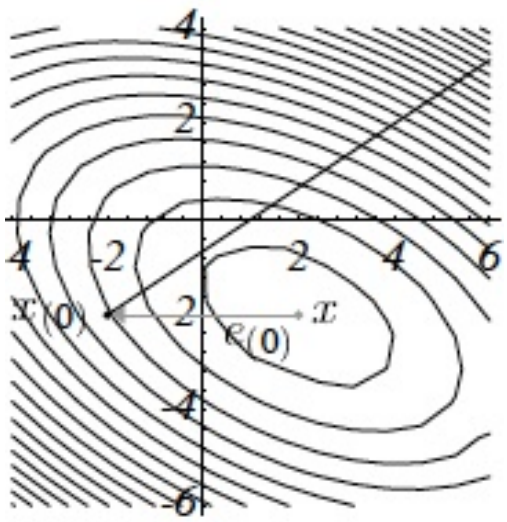
Steepest Descent

- We have a *terrain* and an objective: get to the lowest point!
- The simplest idea is to select the direction in which $f(x)$ decreases more rapidly

  this is obviously the direction opposite to $f'(x)$
- Recall the residual: $b - Ax^p = \rho^p$ and the fact that $f'(x) = Ax - b$

- A steepest descent iterate will be $x^{p+1} = x^p + \alpha\rho_p$
- How do we stop?
- How do we define α?



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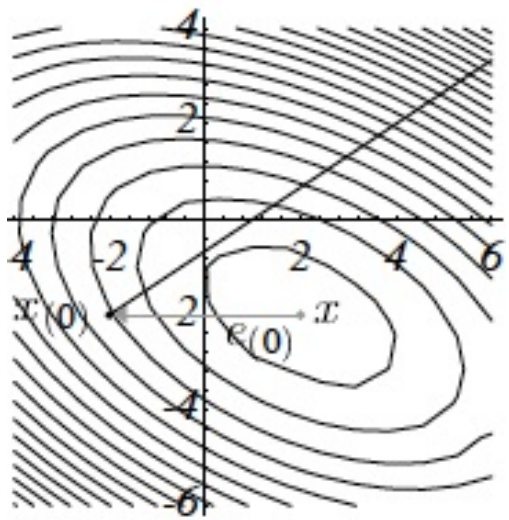

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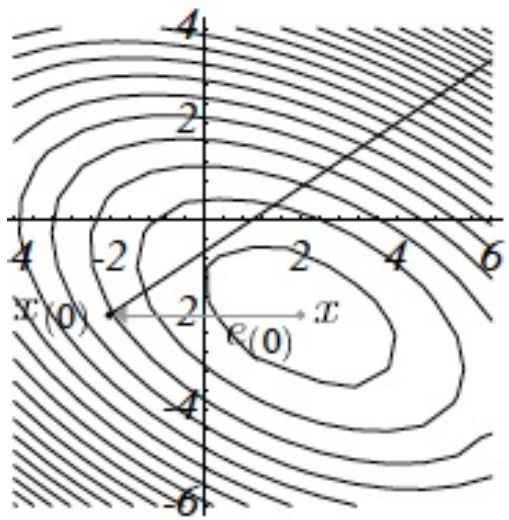

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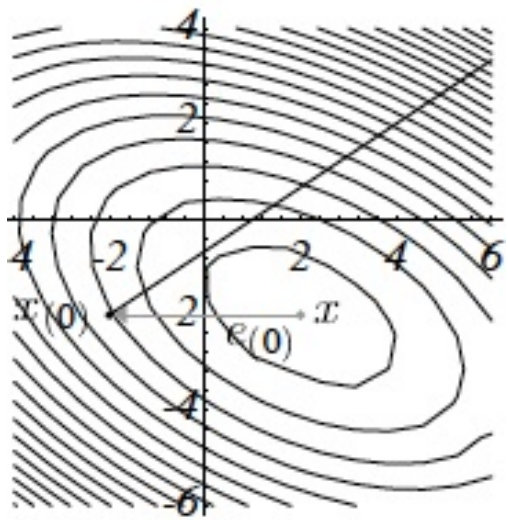

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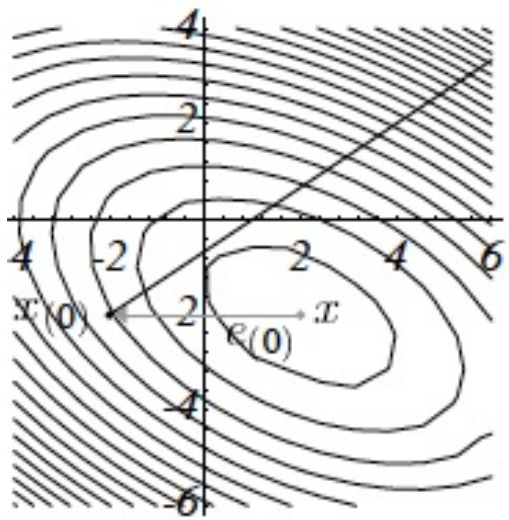

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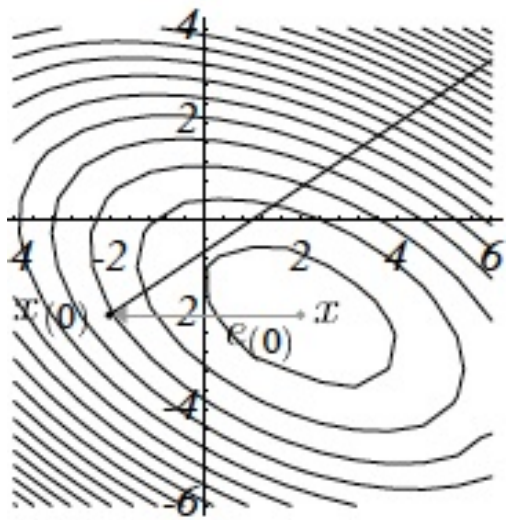

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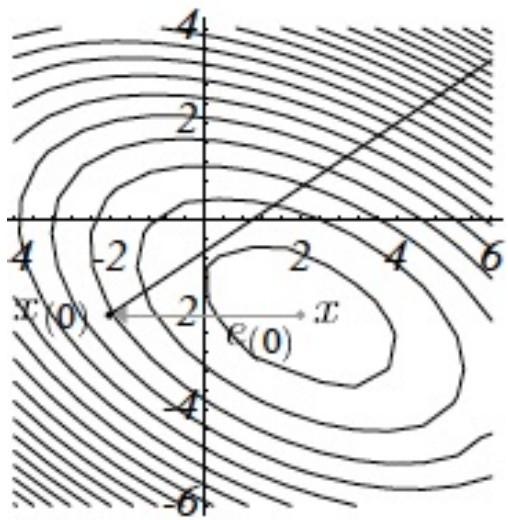

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- Iterate is $x^{p+1} = x^p + \alpha\rho_p$
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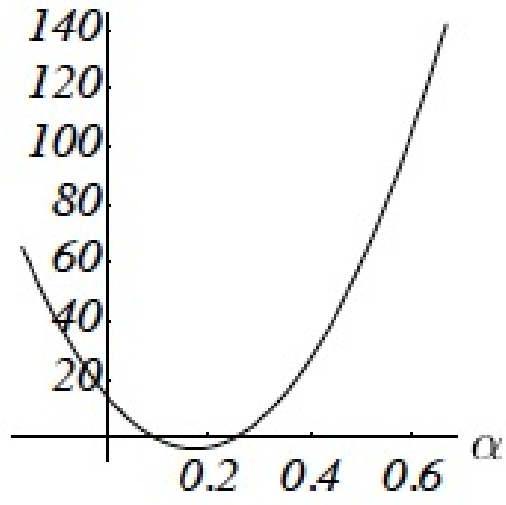

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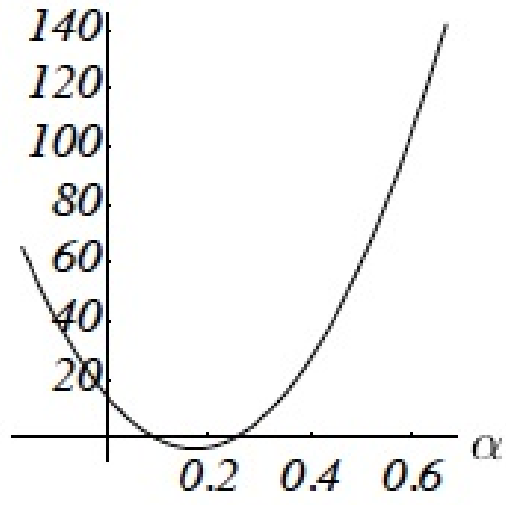

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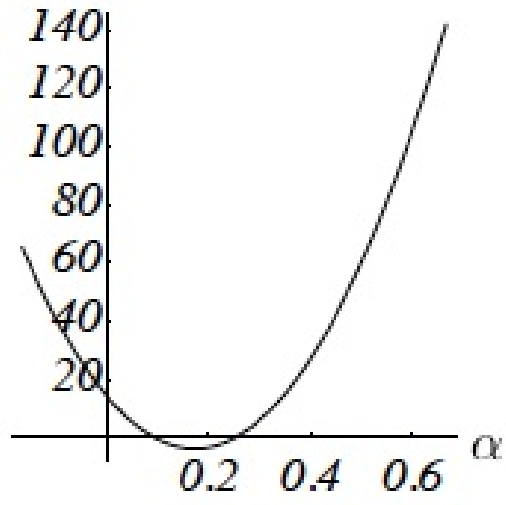

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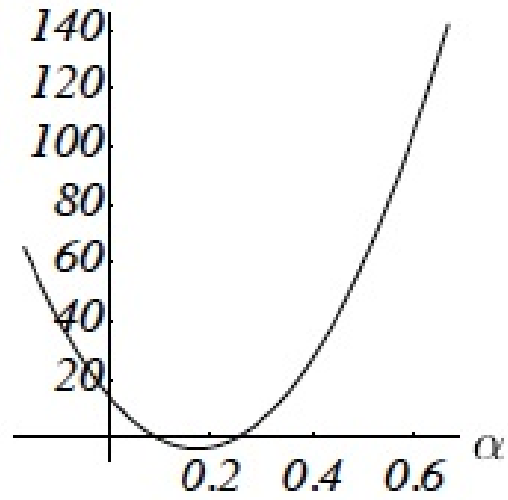

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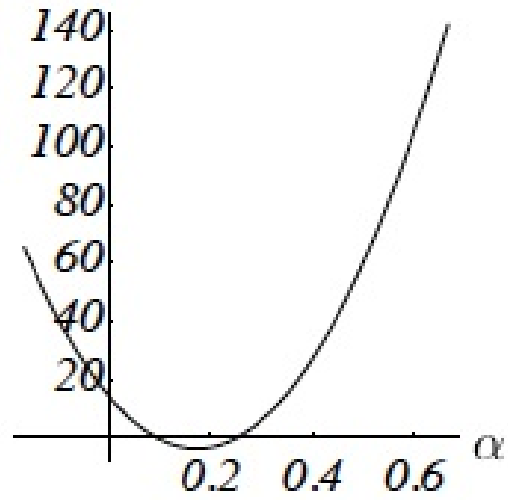

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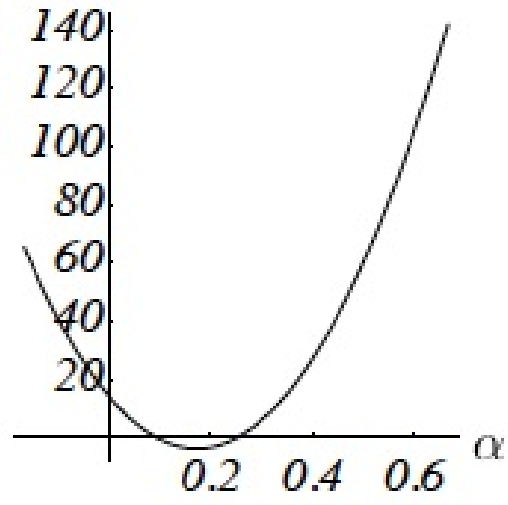

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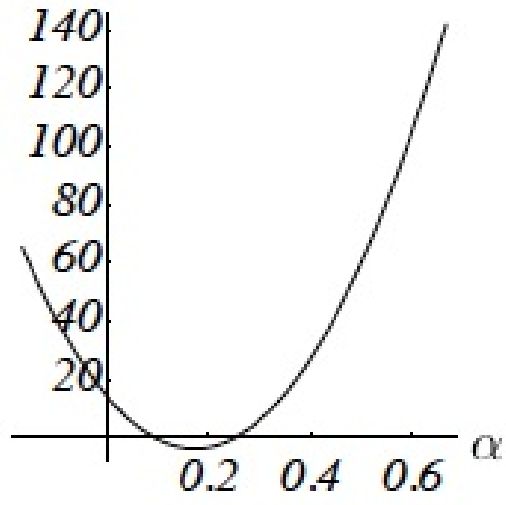

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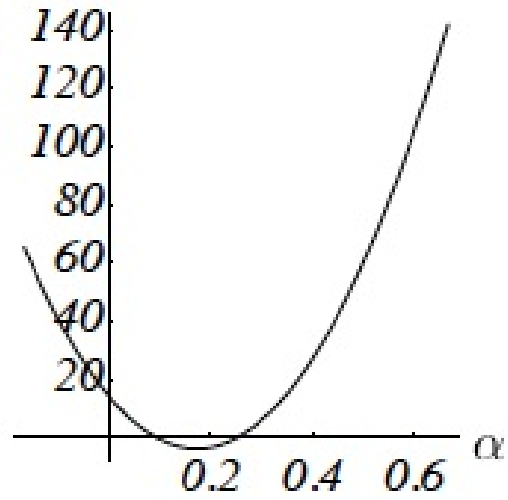

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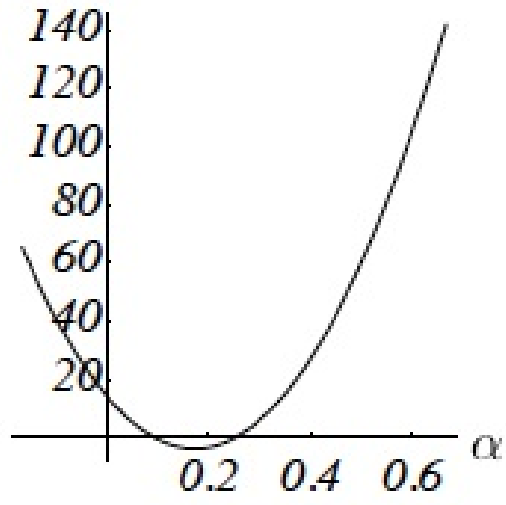

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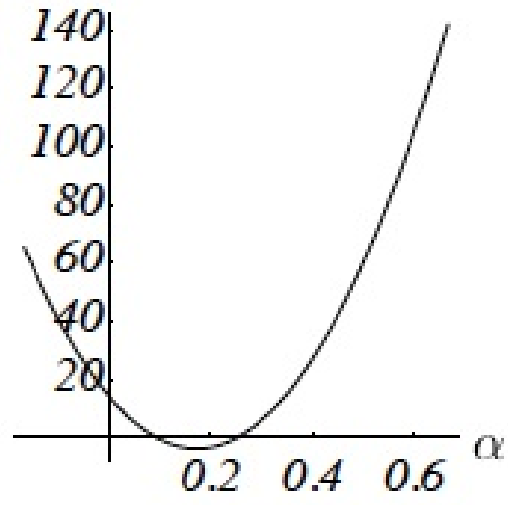

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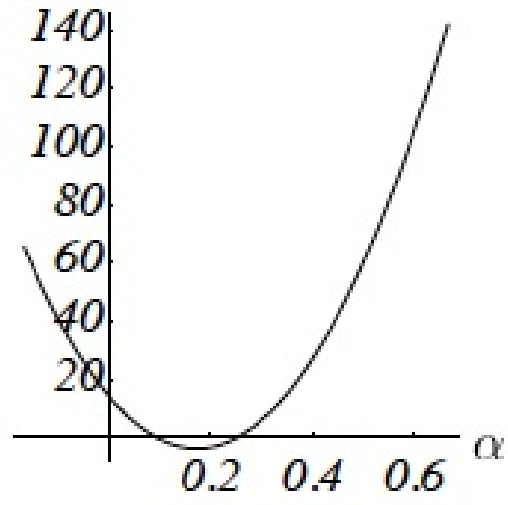

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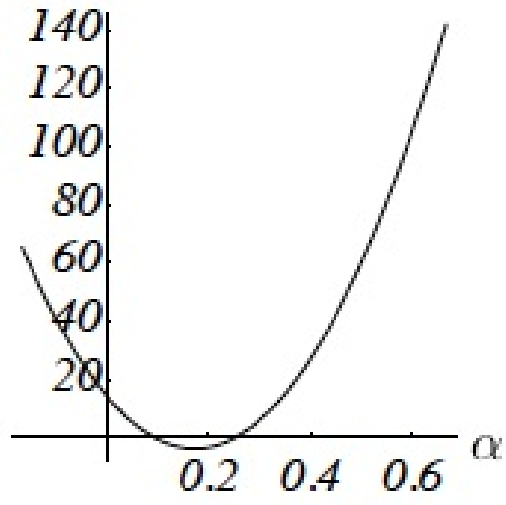
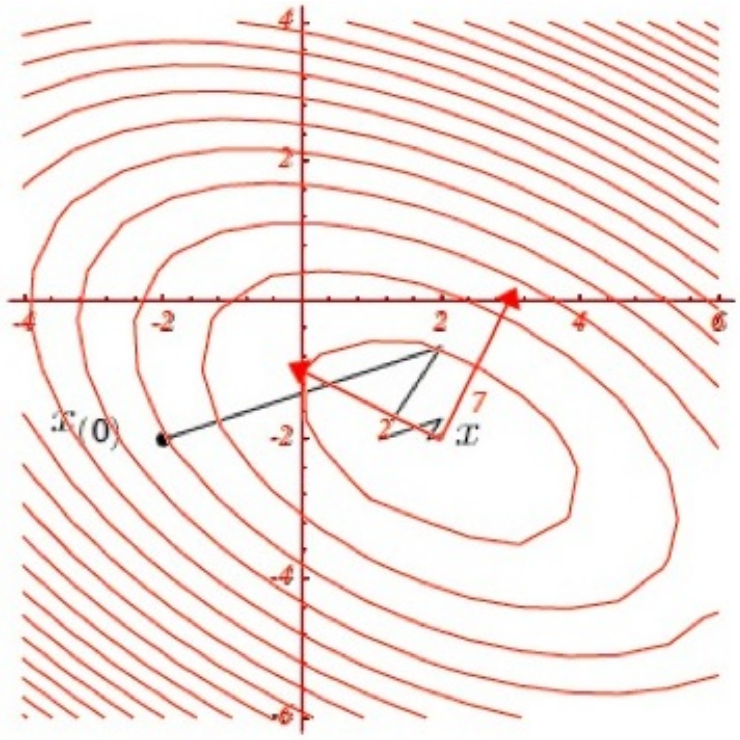
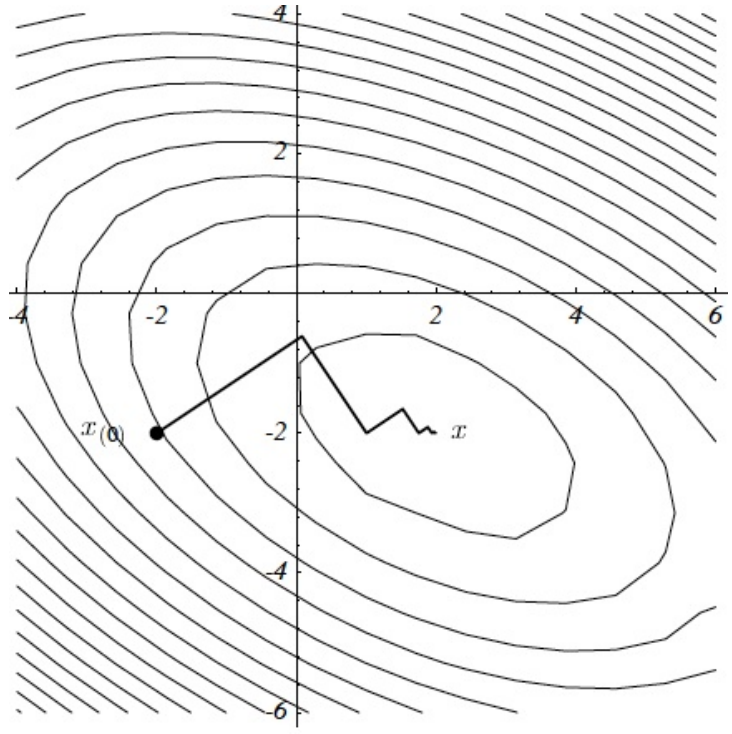


Steepest Descent

- Final iterate is $x^{p+1} = x^p + \alpha_p \rho_p$
- How does the convergence compare to the Jacobi iterate?



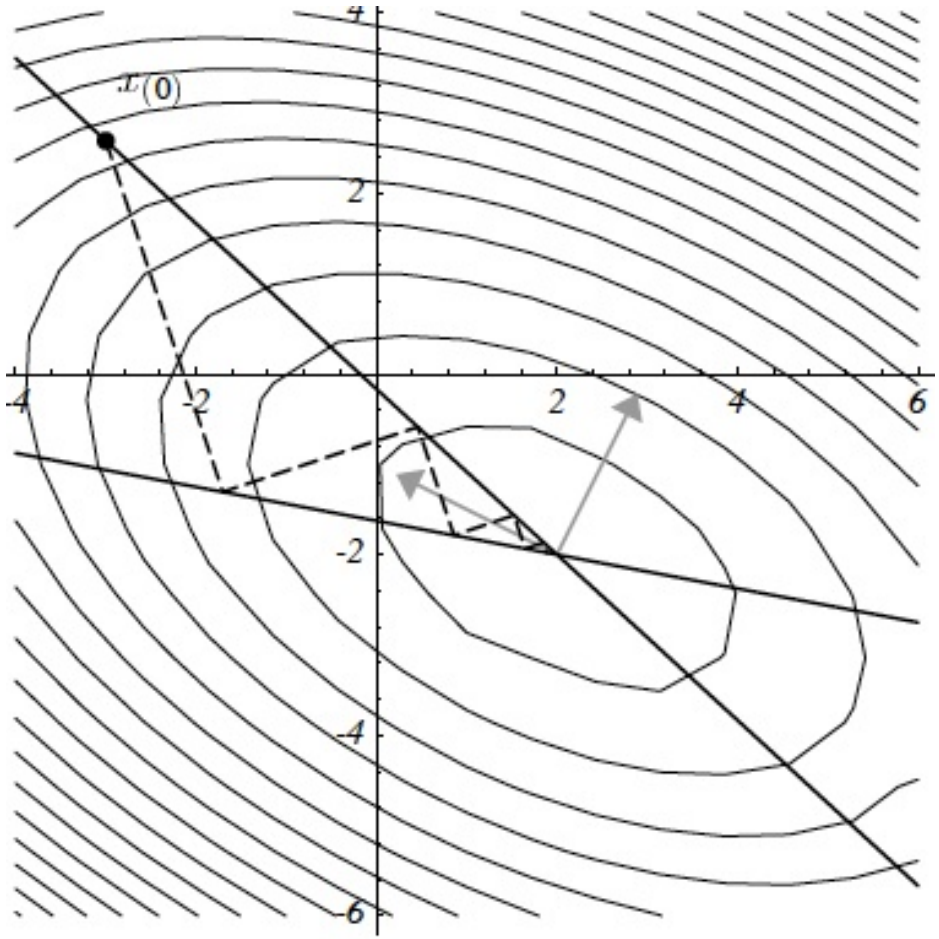
Jacobi



Steepest Descent

Generalized Descent Methods

A pathological case for the steepest descent



The directions are uncorrelated!

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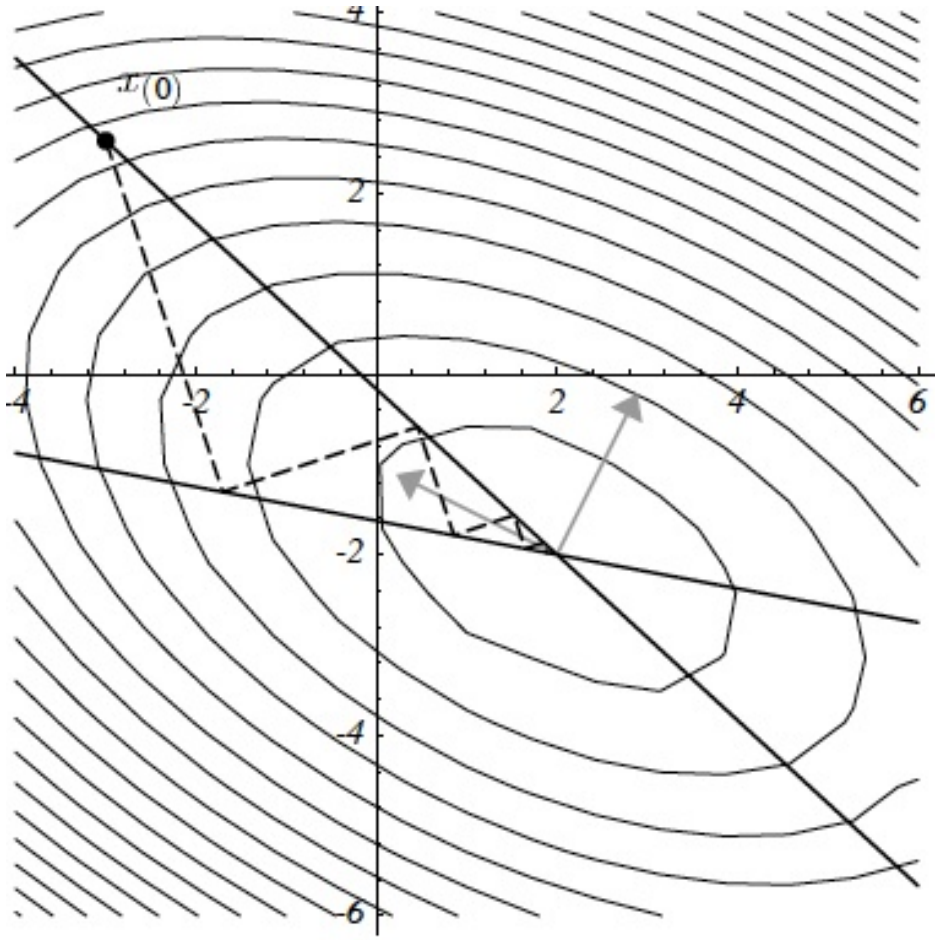

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Conjugate Gradient