Multivariate Polynomials & Sparse Grids

- In UQ, polynomial methods are often used to construct surrogates of (stochastic) parameter dependent quantities of interest (QOIs). These surrogates can be used to cheaply estimate statistics of the QOIs that are potentially expensive to evaluate.
- QOIs may depend on multiple parameters, so we need multivariate polynomial approximations.
- *How do we construct multivariate polynomial approximations? When are they appropriate? When do they fail?*
- Multivariate polynomial approximations suffer from the so-called *curse of dimensionality*, i.e. the work required to construct the approximation grows exponentially with the number of parameters (the dimension).
- We can use *sparse grid collocation* to alleviate (not eliminate) the curse of dimensionality.

Goals for class

- See the connection between tensor product pseudospectral and tensor product spectral collocation.
- Understand difficulty in multivariate approximation with a total order basis.
- Understand the construction of sparse grid collocation.

Notation & Definitions

- Multi-index $\underset{\sim}{i} = (i_1, \dots, i_d) \in \mathbb{N}_0^d$.
- Multi-index sets $\mathcal{I}$. Examples: total order $\{\underset{\sim}{i} \,:\, i_1 + \cdots + i_d \leq n\}$, tensor product $\{\underset{\sim}{i} \,:\, i_k < n_k\}$ for a given multi-index $\underset{\sim}{n} = (n_1, \dots, n_d)$.
- Multivariate orthogonal polynomial $\pi_{\underset{\sim}{i}}(s) = \pi_{i_1}(s_1) \cdots \pi_{i_d}(s_d)$, where each $\pi_{i_k}(s_k)$ is a univariate orthogonal polynomial.
- Multivariate quadrature point $\lambda_{\underset{\sim}{i}} = (\lambda_{i_1}, \dots, \lambda_{i_d})$, where each λ_{i_k} is from a univariate quadrature rule.

- Multivariate quadrature weight $\nu_{\underset{\sim}{i}} = \prod_{i=1}^{d} \nu_{i_k}$, where ν_{i_k} is the weight corresponding to λ_{i_k} from the univariate quadrature rule.