MS&E 112/212: Combinatorial Optimization

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HW#2 – Due 2/14/2017

Homework drop-off: you should deposit your homework in MS&E212 drop-off box located in Huang basement before 5:00 pm on the due date.
We do not collect any homework in class.

Collaboration policy: you can only collaborate with ONE registered student. You may not discuss the homework (this includes clarifications,solutions, etc) with anyone but the instructor, the course assistant or the person you picked. Please write the name of your collaborator in your homework.

1. Recall the definition of a bipartite graph. Let $G(V, E)$ be a graph and (A, B) be a partition of V. We say that G is bipartite if all edges in E have one end-point in A and the other in B. More precisely, for all $(u, v) \in E$ either $u \in A, v \in B$ or $u \in B, v \in A$.

   (a) Prove that a graph is bipartite if and only if it doesn't have an odd cycle.

   (b) A graph is called k-regular if all vertices have degree k. Prove that if a bipartite G is also k-regular with $k \geq 1$ then $|A| = |B|$.

2. **Kleinberg and Tardos 7.9** Network flow issues come up in dealing with natural disasters and other crises, since major unexpected events often require the movement and evacuation of large numbers of people in a short amount of time.

   Consider the following scenario. Due to large-scale flooding in a region, paramedics have identified a set of n injured people distributed across the region who need to be rushed to hospitals. There are k hospitals in the region, and each of the n people needs to be brought to a hospital that is within a half-hour's driving time of their current location (so different people will have different options for hospitals, depending on where they are right now).

   At the same time, one doesn't want to overload any one of the hospitals by sending too many patients its way. The paramedics are in touch by cell phone, and they want to collectively work out whether they can choose a hospital for each of the injured people in such a way that the load on the hospitals is *balanced*: Each hospital receives at most $\lceil n/k \rceil$ people.

   Use maximum-flow to design an algorithm that takes the given information about the people's locations and determines whether this is possible.

3. Given a list of personnel (n persons) and of list of k vacation periods, each period spanning several contiguous vacation days. Let D_j be the set of days included in the jth vacation period. You need to produce a schedule satisfying:

   - For a given parameter c, each tech support person should be assigned to work at most c vacation days total.
   - For each vacation period j, each person should be assigned to work at most one of the days during the period.

- Each vacation day should be assigned a single tech support person.
- For each person, only certain vacation periods are viable.

Describe a polynomial time algorithm to generate an assignment or output that no assignment exists.

4. **Cook, Cunningham, Pulleyblank, and Schrijver 2.35** Suppose that we are given tasks $t_1, t_2, \ldots, t_k$. Each task t_i has a processing time p_i. For certain pairs (i, j), t_i *must precede* t_j, that is, the processing of t_j cannot begin until the processing of t_i is completed. We wish to schedule the processing of the tasks so that all of the tasks are completed as soon as possible. Show how this problem can be solved using a shortest path algorithm.