Homework drop-off: Please bring homework to class to turn in on due date.

Collaboration policy: you can only collaborate with ONE registered student. You may not discuss the homework (this includes clarifications, solutions, etc) with anyone but the instructor, the course assistant or the person you picked. Please write the name of your collaborator in your homework.

1. Show that, in any group of two or more people, there are always two with exactly the same number of friends inside the group. Assume friendship is symmetric i.e. if \( a \) is a friend of \( b \), \( b \) is also a friend of \( a \).

2. [Lovasz, Pelikan, and Vesztergombi 8.5.10] If \( C \) is a cycle, and \( e \) is an edge connecting two nonadjacent nodes of \( C \), then we call \( e \) a chord of \( C \). Prove that if every node of a graph has degree at least 3, then \( G \) contains a cycle with a chord.

3. A double star is a tree that has exactly two nodes that are not leaves. How many labeled double stars are there on \( n \) nodes?

4. [Kleinberg and Tardos 4.8] Suppose you are given a connected graph \( G \), with edge costs that are all distinct. Give an algorithm for finding a spanning tree that has the second smallest cost.

5. The diameter of \( G \) is the maximum distance between two vertices of \( G \), where the distance between two vertices \( u \) and \( v \) in \( G \) is the length of the shortest \( (u, v) \)-path in \( G \) (or infinite if no path exists). Show that if \( G \) has diameter greater than three, then \( \bar{G} \) has diameter less than three. Note that \( \bar{G} \) is the complement graph of \( G \); it is defined on the same set of vertices and an edge \((u, v)\) exists in \( \bar{G} \) if it does not exist in \( G \).

6. [Cayley’s theorem using linear algebra (extra credit)] Recall that Cayley’s theorem states that the number of labeled trees on \( n \) vertices is \( n^{n-2} \).

(a) The unoriented incidence matrix (or simply incidence matrix) of an undirected graph with \( n \) vertices and \( m \) edges is a \( n \times m \) matrix \( B \), such that \( B_{i,j} = 1 \) if the vertex \( v_i \) and edge \( e_j \) are incident and 0 otherwise. The oriented incidence matrix of an undirected graph is the incidence matrix, in the sense of directed graphs, of any orientation of the graph. That is, in the column of edge \( e \), there is one 1 in the row corresponding to one vertex of \( e \) and one −1 in the row corresponding to the other vertex of \( e \), and all other rows have 0.

Let \( B \) be an oriented edge incidence matrix of an undirected graph \( G(V, E) \) on \( n \) vertices. Let \( M = BB^T \). Show that for any \( i \in \{1, 2, \ldots, n\} \),

\[
\det M_{ii} = \sum_N (\det N)^2,
\]
where \( M_{ii} = M\setminus\{i\text{ th row and column}\} \), and \( N \) runs over all \((n - 1) \times (n - 1)\) submatrices of \( B\setminus\{i\text{ th row}\} \). Each submatrix \( N \) corresponds to a choice of \( n - 1 \) edges of \( G \).

(b) Show that
\[
\det N = \begin{cases} 
\pm 1 & \text{if edges form a tree} \\
0 & \text{otherwise} 
\end{cases}
\]
This implies that \( t(G) = \det M_{ii} \), where \( t(G) \) is the number of spanning trees of \( G \).

(c) Show that for the complete graph on \( n \) vertices \( K_n \),
\[
\det M_{ii} = n^{n-2}.
\]
Conclude that this implies Cayley theorem.