MS&E 112/212: Combinatorial Optimization
Instructor: Professor Amin Saberi (saberi@stanford.edu)
HW4 – Due 3/14/2017

Homework drop-off: you should turn in your homework in class on the due date or email it to Nolan.

Collaboration policy: you can only collaborate with ONE registered student. You may not discuss the homework (this includes clarifications,solutions, etc) with anyone but the instructor, the course assistant or the person you picked. Please write the name of your collaborator in your homework.

1. **The stable roommate problem.** The stable roommate problem is similar to the stable marriage problem, except pairings are made within a single pool rather than between two genders. We define a rogue pair with respect to a matching M as a pair (a, b) such that a prefers b to his current partner in M and b prefers a to his current partner in M. A stable pairing is a matching that has no rogue pairs. Give an example of the stable roommate problem in which no stable pairing exists.

2. Given a *directed* graph $G(V, E)$ we define edge weights $w_{ij} > 0$ whenever $(i, j) \in E$ and $w_{ij} = 0$ otherwise. Given a set $S \subseteq V$, the cut value obtained by partitioning the vertices into S and its complement $\bar{S}$ is given by

$$\text{cut}(S, \bar{S}) = \sum_{i \in S, j \in \bar{S}} w_{ij}$$

   The directed max-cut problem is to find S that maximizes $\text{cut}(S, \bar{S})$.

   Give a randomized algorithm to solve directed max-cut that has an approximation ratio of 1/4 in expectation.

3. **(Kleinberg Tardos 11.10)** Suppose you are given an n by n grid graph G with vertex weights $w(v) \geq 0$ that are all distinct and integer. The goal is to choose an independent set S of nodes of the grid, so that the sum of the weights of the nodes in S is as large as possible.

   Consider the "heaviest-first" greedy algorithm. Start with $S = \emptyset$, and while $|V| \neq 0$ pick the node $v_i \in V$ of maximum weight, add v_i to S and delete its neighbors from G. The resulting S will be an independent set by construction.

   (a) Let S be the independent set returned by the algorithm above, and let T be any other independent set in G. Show that, for each node $v \in T$, either $v \in S$, or there is a node $v' \in S$ so that $w(v) \leq w(v')$ and $(v, v') \in E$.

   (b) Show that this algorithm returns an independent set of total weight at least $\frac{1}{4}$ times the maximum total weight of any independent set in the grid graph G.