Assignment #2 (Due April 26)

1. Consider the distribution function

\[
F(x) = \begin{cases} 
0 & \text{for } x < -1 \\
1/4 & \text{for } -1 \leq x < 1 \\
(x+1)^2/12 & \text{for } 1 \leq x < 2 \\
(x+4)/8 & \text{for } 2 \leq x < 4 \\
1 & \text{for } x \geq 4 
\end{cases}
\]

a) Assuming that a uniform random number generator is available, use the inversion method to provide an algorithm for generating a random variable \(X\) having distribution function \(F\).

b) Compute \(E[X]\) and \(\text{Var}[X]\) analytically.

2. (Loading Dock) A warehouse contains \(N\) items that need to be shipped out on a given day. There are \(N\) trucks available to transport the items, and each truck carries exactly one item. (The items are identical, so any truck can transport any item.) The warehouse has a loading dock that can accommodate up to \(N\) trucks simultaneously. The warehouse starts operations at time 0, at which time the \(N\) trucks start heading toward the loading dock. The time for a truck to reach the loading dock is exponentially distributed with an expected value of \(\mu_1\) time units, independently of the other trucks. Also at time 0, a conveyor starts to deliver items, one at a time, from inside the warehouse to the loading dock. The successive times between arrivals of items at the loading dock are independent and exponentially distributed with an expected value of \(\mu_2\) time units. Trucks load items on a first-come, first-served basis. Assume that the time for the truck to load an item is negligible, and that the truck departs the instant that it is loaded.

Set \(X(t) = (M(t), N(t))\), where \(N(t)\) is the number of trucks that arrive at the loading dock and \(M(t)\) is the number of items that arrive on the loading dock during the interval \([0,t]\). It is not hard to see that \(\{X(t) : t \geq 0\}\) is a CTMC.

a) Specify the intensity vector \(q\) and the rate matrix \(Q\) for the CTMC in terms of \(\mu_1\) and \(\mu_2\). I.e., give general formulas for \(q((m,n))\) and \(Q((m,n),(m',n'))\).

b) In terms of the process \(\{X(t) : t \geq 0\}\), give a precise specification of (i) the expected time until all trucks have departed, (ii) the expected average delay for all of the trucks, where a delay is the time from when a truck arrives at the loading dock until it departs, and (iii) the probability that there are ever more than 5 trucks simultaneously waiting at the loading dock. [Hint: When defining some of these quantities, you may wish to define some intermediate random variables. It may be useful to use the notation \(X(t-)\) to denote the state of the process just before time \(t\).]
c) Suppose that $N = 8$, $\mu_1 = 1/2$ hour, and $\mu_2 = 20$ minutes. Using the general algorithm given in class for simulation of CTMCs, write a program to simulate the CTMC $\{X(t): t \geq 0\}$ and estimate each of the performance measures in part (b) to within $\pm 5\%$ with probability 99\%.