Agenda and key issues

• Pricing with binomial trees
  – Replication
  – Risk-neutral pricing
• Interest rate models
  – Definitions
  – Uses
  – Features
  – Implementation
• Binomial tree example
• Embedded options
  – Callable bond
  – Putable bond
• Factor models
  – Spot rate process
  – Drift and volatility functions
  – Calibration
1-period binomial model

• Stock with price $S = 60$ and one-period risk-free rate of $r = 20\%$

• Over next period stock price either falls to $30$ or rises to $90$
  \[ S_d = 30 \]
  \[ S_u = 90 \]

• Call option with strike price $K = 60$ pays either $0$ or $30$
  \[ C_u = 30 \]
  \[ C_d = 0 \]

• Buy $\Delta = \frac{1}{2}$ share of stock and borrow $L = 12.50$
  \[ \frac{S_u}{2} - 1.2 \times 12.5 = 45 - 15 = 30 \]
  \[ \frac{S_d}{2} - 1.2 \times 12.5 = 15 - 15 = 0 \]
1-period binomial model (cont)

- Portfolio replicates option payoff \( \Rightarrow C = $17.50 \)

- Solving for replicating portfolio
  - Buy \( \Delta \) shares of stock and borrow \( L \)
  - If stock price rises to $90, we want the portfolio to be worth
    \[ 90 \times \Delta - 1.2 \times L = $30 \]
  - If stock price drops to $30, we want the portfolio to be worth
    \[ 30 \times \Delta - 1.2 \times L = $0 \]
  - \( \Delta = 0.5 \) and \( L = $12.50 \) solve these two equations
1-period binomial model (cont)

• “Delta”
  – $\Delta$ is chosen so that the value of the replicating portfolio ($\Delta \times S - L$) has the same sensitivity to $S$ as the option price $C$
    
    $\Delta = \frac{dC}{dS} = \frac{\$30 - \$0}{\$90 - \$30} = \frac{1}{2}$

  – $\Delta$ is called the hedge ratio of “delta” of the option
  – Delta-hedging an option is analogous to duration-hedging a bond
1-period binomial model (cont)

• Very important result
  The option price does not depend on the probabilities of a stock price up-move or down-move

• Intuition
  – If \( C \neq \Delta \times (S - L) \), there exist an arbitrage opportunity
  – Arbitrage opportunities deliver riskless profits
  – Riskless profits cannot depend on probabilities
  – Therefore, the option price cannot depend on probabilities
1-period binomial model (cont)

- Unfortunately, this simple replication argument does not work with 3 or more payoff states

\[
\begin{align*}
S & \quad S_u \quad S_m \quad S_d \quad C = ? \quad C_u \quad C_m \quad C_d \\
\end{align*}
\]

- Rather than increase the number of payoff states per period, increase the number of binomial periods ⇒ binomial tree

Recombining

Non-recombining
1-period binomial model (cont)

- Define
  - \( u = 1 + \) return if stock price goes up
  - \( d = 1 + \) return if stock price goes down
  - \( r = \) per-period riskless rate (constant for now)
  - \( p = \) probability of stock price up-move

- No arbitrage requires \( d \leq 1 + r \leq u \)

- Stock and option payoffs

\[
\begin{align*}
S & \quad S \times u & S \times d \\
C & = ? & C_u = f(S \times u) & C_d = f(S \times d)
\end{align*}
\]
1-period binomial model (cont)

- Payoff of portfolio of \( \Delta \) shares and \( L \) dollars of borrowing
  \[
  \Delta S - L < \Delta S \times d - L \times (1 + r)
  \]

- Replication requires
  \[
  \Delta \times S \times u - L \times (1 + r) = C_u
  \]
  \[
  \Delta \times S \times d - L \times (1 + r) = C_d
  \]

- Two equations in two unknowns (\( \Delta \) and \( L \)) with solution
  \[
  \Delta = \frac{C_u - C_d}{S \times (u - d)}
  \]
  \[
  L = \frac{d \times C_u - u \times C_d}{(1 + r) \times (r - d)}
  \]

- Option price
  \[
  C = \Delta \times S - L
  \]
Risk-neutral pricing (cont)

• Define
  
  \[ q = \frac{(1 + r) - d}{u - d} \quad (1 - q) = \frac{u - (1 + r)}{u - d} \]

• No-arbitrage condition \( d \leq 1 + r \leq u \) implied \( 0 \leq q \leq 1 \)

• Rearrange option price

  \[ C = \Delta \times S - L \]

  \[ = \frac{C_u - C_d}{S \times (u - d)} \times S - \frac{d \times C_u - u \times C_d}{(1 + r) \times (u - d)} \]

  \[ = \frac{q \times C_u + (1 - q) \times C_d}{1 + r} \]
Risk-neutral pricing (cont)

• Interpretation of $q$
  – Expected return on the stock
    \[
    E\left[ \frac{S_1}{S_0} \right] = \frac{p \times S \times u + (1 - p) \times S \times d}{S} = p \times u + (1 - p) \times d
    \]
  – Suppose we were risk-neutral
    \[
    E\left[ \frac{S_1}{S_0} \right] = p \times u + (1 - p) \times d = (1 + r)
    \]
  – Solving for $p$
    \[
    p = \frac{(1 + r) - d}{u - d} = q
    \]

• Very, very important result
  $q$ is the probability which sets the expected return on the stock equal to the riskfree rate ⇒ risk-neutral probability
Risk-neutral pricing (cont)

• Very, very, very important result

The option price equals its expected payoff discounted by the riskfree rate, where the expectation is formed using risk-neutral probabilities instead of real probabilities ⇒ risk-neutral pricing

• Risk-neutral pricing extends to multiperiod binomial trees and applies to all derivatives which can be replicated

\[
\text{Derivatives price} = PV_r \left[ E^q [\text{payoff}] \right]
\]
Risk-neutral pricing intuition

• **Step 1**
  – Derivatives are priced by no-arbitrage
  – No-arbitrage does not depend on risk preferences or probabilities

• **Step 2**
  – Imagine a world in which all security prices are the same as in the real world but everyone is risk-neutral (a “risk-neutral world”)
  – The expected return on any security equals the risk-free rate $r$

• **Step 3**
  – In the risk-neutral world, every security is priced as its expected payoff discounted by the risk-free rate, including derivatives
  – Expectations are taken wrt the risk-neutral probabilities $q$

• **Step 4**
  – Derivative prices must be the same in the risk-neutral and real worlds because there is only one no-arbitrage price
### 2-period binomial model

- **Stock and option payoffs**

- **By risk-neutral pricing**

\[
C = \frac{q^2 \times C_{uu} + 2 \times q \times (1 - q) \times C_{ud} + (1 - q)^2 \times C_{dd}}{(1 + r)^2}
\]
3-period binomial model

- Stock and option payoffs

- By risk-neutral pricing

\[
C = \frac{1}{(1 + r)^3} \times \left[ q^3 \times C_{uuu} + 3 \times q^2 \times (1 - q) \times C_{uud} + \ldots + 3 \times q \times (1 - q)^2 \times C_{udd} + (1 - q)^3 \times C_{ddd} \right]
\]

\[
C_{uu} = f(S \times u^3)
\]

\[
C_{u} = f(S \times u^2)
\]

\[
C_{d} = f(S \times d)
\]

\[
C_{ud} = f(S \times u \times d)
\]

\[
C_{udd} = f(S \times u \times d^2)
\]

\[
C_{ddd} = f(S \times d^3)
\]
Definitions

• An interest rate model describes the dynamics of either
  – 1-period spot rate
  – Instantaneous spot rate = \( t \)-year spot rate \( r(t) \) as \( t \to 0 \)

• Variation in spot rates is generated by either
  – One source of risk ⇒ single-factor models
  – Two or more sources of risk ⇒ multifactor models
Model uses

• Characterize term structure of spot rates to price bonds

• Price interest rate and bond derivatives
  – Exchange traded (e.g., Treasury bond or Eurodollar options)
  – OTC (e.g., caps, floors, collars, swaps, swaptions, exotics)

• Price fixed income securities with embedded options
  – Callable or putable bonds

• Compute price sensitivities to underlying risk factor(s)

• Describe risk-reward trade-off
Model features

• Interest rate models should be
  – Arbitrage free = model prices agree with current market prices
    ▪ Spot rate curve
    ▪ Coupon yield curve
    ▪ Interest rate and bond derivatives
  – Time-consistent = model implied behavior of spot rates and bond prices agree with their observed behavior
    ▪ Mean reversion
    ▪ Conditional heteroskedasticity
    ▪ Term structure of volatility and correlation structure

• Developing an interest rate model which is both arbitrage free and time consistent is the holy grail of fixed income research
Model implementation

• In practice, two model implementations
  – Cross-sectional calibration
    ▪ Calibrate model to match exactly all market prices of liquid securities on a single day
    ▪ Used for pricing less liquid securities and derivatives
    ▪ Arbitrage free but probably not time-consistent
    ▪ Usually one or two factors
  – Time-series estimation
    ▪ Estimate model using a long time-series of spot rates
    ▪ Used for hedging and asset allocation
    ▪ Time-consistent but not arbitrage free
    ▪ Usually two and more factors
Spot rate tree

• 1-period spot rates ($m$-period compounded APR)

\[ r_{0,0}(1) = 10\% \]
\[ r_{1,0}(1) = 9\% \]
\[ r_{1,1}(1) = 11\% \]
\[ r_{2,0}(1) = 8\% \]
\[ r_{2,1}(1) = 10\% \]
\[ r_{2,2}(1) = 12\% \]

• Notation
  
  – $r_{ij}(n)$ = $n$-period spot rate $i$ periods in the future after $j$ up-moves
  
  – $\Delta t$ = length of a binomial step in units of years

• Set $\Delta t = 1/m$ and $m = 2$

• Assume $q_{ij} = 0.5$ for all steps $i$ and nodes $j$
Road-map

• Calculate step-by-step
  – Implied spot rate curve $r_{0,0}(1)$, $r_{0,0}(2)$, $r_{0,0}(3)$
  – Implied changes in the spot rate curve

  \[
  r_{0,0}(1), r_{0,0}(2) \quad \quad \quad \quad \quad \quad \quad r_{1,1}(1), r_{1,1}(2)
  \]
  \[
  \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad r_{1,0}(1), r_{1,0}(2)
  \]

  – Price 8% 1.5-yr coupon bond
  – Price 1-yr European call option on 8% 1.5-yr coupon bond
  – Price 1-yr American put option on 8% 1.5-yr coupon bond
1-period zero-coupon bond prices

- At time 0

\[ P_{0,0}(1) = ? \]

\[ P_{1,1}(0) = 100 \]

\[ P_{1,0}(0) = 100 \]

\[
P_{0,0}(1) = \frac{q_{0,0} \times P_{1,1}(0) + (1 - q_{0,0}) \times P_{1,0}(0)}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}}
\]

\[
= \frac{100}{(1 + 0.1/2)^1} = 95.24
\]
1-period zero-coupon bond prices (cont)

• At time 1

\[ P_{1,1}(1) = \frac{q_{1,1} \times P_{2,2}(0) + (1 - q_{1,1}) \times P_{2,1}(0)}{(1 + r_{1,1}(1)/m)^{1 \times \Delta t \times m}} \]

\[ = \frac{$100}{(1 + 0.11/2)^1} = $94.79 \]

\[ P_{1,0}(1) = \frac{q_{1,0} \times P_{2,1}(0) + (1 - q_{1,0}) \times P_{2,0}(0)}{(1 + r_{1,0}(1)/m)^{1 \times \Delta t \times m}} \]

\[ = \frac{$100}{(1 + 0.09/2)^1} = $95.69 \]
1-period zero-coupon bond prices (cont)

• At time 2

\[
P_{2,2}(1) = \frac{q_{2,2} \times P_{3,3}(0) + (1 - q_{2,2}) \times P_{3,2}(0)}{(1 + r_{2,2}(1)/m)^{1\times\Delta t\times m}}
\]

\[
= \frac{$100}{(1 + 0.12/2)^1} = $94.34
\]

\[
P_{2,1}(1) = \frac{q_{2,1} \times P_{3,2}(0) + (1 - q_{2,1}) \times P_{3,1}(0)}{(1 + r_{2,1}(1)/m)^{1\times\Delta t\times m}} = $95.24
\]

\[
P_{2,0}(1) = \frac{q_{2,0} \times P_{3,1}(0) + (1 - q_{2,0}) \times P_{3,0}(0)}{(1 + r_{2,0}(1)/m)^{1\times\Delta t\times m}}
\]

\[
= \frac{$100}{(1 + 0.08/2)^1} = $96.15
\]
1-period zero-coupon bond prices (cont)

\[ P_{0,0}(1) = $95.24 \]
\[ P_{1,0}(1) = $95.69 \]
\[ P_{1,1}(1) = $94.79 \]
\[ P_{2,0}(1) = $96.15 \]
\[ P_{2,1}(1) = $95.25 \]
\[ P_{2,2}(1) = $94.34 \]
2-period zero-coupon bond prices

- At time 0

\[ P_{1,1}(1) = 94.79 \]

\[ P_{1,0}(1) = 95.69 \]

\[ P_{0,0}(2) = ? \]

\[ P_{0,0}(2) = \frac{q_{0,0} \times P_{1,1}(1) + (1 - q_{0,0}) \times P_{1,0}(1)}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}} \]

\[ = \frac{1}{2} \times 94.79 + \frac{1}{2} \times 95.69 \]

\[ = \frac{1}{(1 + 0.1/2)^1} = 90.71 \]

- Implied 2-period spot rate

\[ P_{0,0}(2) = \frac{\$100}{(1 + r_{0,0}(2)/m)^{2 \times \Delta t \times m}} \Rightarrow r_{0,0}(2) = 9.9976\% \]
2-period zero-coupon bond prices (cont)

- At time 1

\[ P_{1,1}(2) = \frac{q_{1,1} \times P_{2,2}(1) + (1 - q_{1,1}) \times P_{2,1}(1)}{(1 + r_{1,1}(1)/m)^{1 \times \Delta t \times m}} \]

\[ = \frac{\frac{1}{2} \times 94.34 + \frac{1}{2} \times 95.24}{(1 + 0.11/2)^1} \]

\[ = 89.85 \]

\[ \Rightarrow r_{1,1}(2) = 10.9976\% \]

\[ P_{1,0}(2) = \frac{q_{1,0} \times P_{2,1}(1) + (1 - q_{1,0}) \times P_{2,0}(1)}{(1 + r_{1,0}(1)/m)^{1 \times \Delta t \times m}} \]

\[ = \frac{\frac{1}{2} \times 95.24 + \frac{1}{2} \times 96.15}{(1 + 0.09/2)^1} \]

\[ = 91.58 \]

\[ \Rightarrow r_{1,0}(2) = 8.9976\% \]
3-period zero-coupon bond price

- At time 0
  \[ P_{1,1}(2) = \$89.85 \]
  \[ P_{1,0}(2) = \$91.58 \]

  \[ P_{0,0}(3) = \ ? \]

  \[ P_{0,0}(3) = \frac{q_{0,0} \times P_{1,1}(2) + (1 - q_{0,0}) \times P_{1,0}(2)}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}} \]
  \[ = \frac{\frac{1}{2} \times \$89.85 + \frac{1}{2} \times \$91.58}{(1 + 0.1/2)^1} = \$86.39 \]

- Implied 3-period spot rate

  \[ P_{0,0}(3) = \frac{\$100}{(1 + r_{0,0}(3)/m)^{3 \times \Delta t \times m}} \Rightarrow r_{0,0}(3) = 9.9937\% \]
Implied spot rate curve

• Current spot rate curve is slightly downward sloping

\[ r_{0,0}(1) = 10.0000\% \]
\[ r_{0,0}(2) = 9.9976\% \]
\[ r_{0,0}(3) = 9.9937\% \]

• From one period to the next, the spot rate curve shifts in parallel

\[ r_{1,1}(1) = 11.0000\% \]
\[ r_{1,1}(2) = 10.9976\% \]
\[ r_{0,0}(1) = 10.0000\% \]
\[ r_{0,0}(2) = 9.9976\% \]
\[ r_{1,0}(1) = 9.0000\% \]
\[ r_{1,0}(2) = 8.9976\% \]
**Coupon bond price**

- 8% 1.5-year (3-period) coupon bond with cashflow

```
$0.00  $4.00  $4.00  $4.00  $104.00
    $4.00  $4.00  $104.00
        $4.00  $104.00
            $4.00  $104.00
```

Interest Rate Models
**Coupon bond price (cont)**

- Discounting terminal payoffs by 1 period

\[ P_{0,0} = ? \]
\[ P_{1,0} = ? \]
\[ P_{1,1} = ? \]

\[ P_{2,0} = \frac{104.00}{1.04} = 100 \]
\[ P_{2,1} = \frac{104.00}{1.05} = 99.05 \]
\[ P_{2,2} = \frac{104.00}{1.06} = 98.11 \]

\[ P_{3,0} = 104.00 \]
\[ P_{3,1} = 104.00 \]
\[ P_{3,2} = 104.00 \]
\[ P_{3,3} = \frac{104.00}{1.04} = 100 \]
Coupon bond price (cont)

- By risk-neutral pricing

\[ P_{1,1} = \frac{c + q_{1,1} \times P_{2,2} + (1 - q_{1,1}) \times P_{2,1}}{(1 + r_{1,1}(1)/m)^{1\times\Delta t\times m}} \]

\[ = \frac{4.00 + \frac{1}{2} \times 98.11 + \frac{1}{2} \times 99.05}{1.055} \]

\[ P_{0,0} = ? \]

\[ P_{1,0} = ? \]

\[ P_{2,0} = $104.00/1.04 = $100 \]

\[ P_{2,1} = $104.00/1.05 = $99.05 \]

\[ P_{2,2} = $104.00/1.06 = $98.11 \]

\[ P_{3,0} = $104.00 \]

\[ P_{3,1} = $104.00 \]

\[ P_{3,2} = $104.00 \]

\[ P_{3,3} = $104.00 \]
Coupon bond price (cont)

- By risk-neutral pricing (cont)

\[
P_{0,0} = ?
\]

\[
P_1 = \frac{c + q_{1,0} \times P_{2,1} + (1 - q_{1,0}) \times P_{2,0}}{(1 + r_{1,0}(1)/m)^{1 \times \Delta t \times m}}
\]

\[
P_1 = \frac{\$4.00 + \frac{1}{2} \times \$99.05 + \frac{1}{2} \times \$100.00}{1.045}
\]

\[
P_1 = \$97.23
\]

\[
P_2 = \frac{\$104.00/1.04}{1.05}
\]

\[
P_2 = \$99.05
\]

\[
P_3 = \frac{\$104.00}{1.06}
\]

\[
P_3 = \$98.11
\]

\[
P_4 = \frac{\$104.00/1.04}{1.06}
\]

\[
P_4 = \$98.11
\]

\[
P_5 = \frac{\$104.00}{1.06}
\]

\[
P_5 = \$98.11
\]

\[
P_6 = \frac{\$104.00/1.04}{1.06}
\]

\[
P_6 = \$98.11
\]

\[
P_7 = \frac{\$104.00}{1.06}
\]

\[
P_7 = \$98.11
\]

\[
P_8 = \frac{\$104.00/1.04}{1.06}
\]

\[
P_8 = \$98.11
\]

\[
P_9 = \frac{\$104.00}{1.06}
\]

\[
P_9 = \$98.11
\]

\[
P_{10} = \frac{\$104.00/1.04}{1.06}
\]

\[
P_{10} = \$98.11
\]

\[
P_{11} = \frac{\$104.00}{1.06}
\]

\[
P_{11} = \$98.11
\]

\[
P_{12} = \frac{\$104.00/1.04}{1.06}
\]

\[
P_{12} = \$98.11
\]

\[
P_{13} = \frac{\$104.00}{1.06}
\]

\[
P_{13} = \$98.11
\]

\[
P_{14} = \frac{\$104.00/1.04}{1.06}
\]

\[
P_{14} = \$98.11
\]

\[
P_{15} = \frac{\$104.00}{1.06}
\]

\[
P_{15} = \$98.11
\]

\[
P_{16} = \frac{\$104.00/1.04}{1.06}
\]

\[
P_{16} = \$98.11
\]

\[
P_{17} = \frac{\$104.00}{1.06}
\]

\[
P_{17} = \$98.11
\]

\[
P_{18} = \frac{\$104.00/1.04}{1.06}
\]

\[
P_{18} = \$98.11
\]

\[
P_{19} = \frac{\$104.00}{1.06}
\]

\[
P_{19} = \$98.11
\]

\[
P_{20} = \frac{\$104.00/1.04}{1.06}
\]

\[
P_{20} = \$98.11
\]

\[
P_{21} = \frac{\$104.00}{1.06}
\]

\[
P_{21} = \$98.11
\]

\[
P_{22} = \frac{\$104.00/1.04}{1.06}
\]

\[
P_{22} = \$98.11
\]

\[
P_{23} = \frac{\$104.00}{1.06}
\]

\[
P_{23} = \$98.11
\]

\[
P_{24} = \frac{\$104.00/1.04}{1.06}
\]

\[
P_{24} = \$98.11
\]

\[
P_{25} = \frac{\$104.00}{1.06}
\]

\[
P_{25} = \$98.11
\]

\[
P_{26} = \frac{\$104.00/1.04}{1.06}
\]

\[
P_{26} = \$98.11
\]

\[
P_{27} = \frac{\$104.00}{1.06}
\]

\[
P_{27} = \$98.11
\]

\[
P_{28} = \frac{\$104.00/1.04}{1.06}
\]

\[
P_{28} = \$98.11
\]

\[
P_{29} = \frac{\$104.00}{1.06}
\]

\[
P_{29} = \$98.11
\]

\[
P_{30} = \frac{\$104.00/1.04}{1.06}
\]

\[
P_{30} = \$98.11
\]

\[
P_{31} = \frac{\$104.00}{1.06}
\]

\[
P_{31} = \$98.11
\]

\[
P_{32} = \frac{\$104.00/1.04}{1.06}
\]

\[
P_{32} = \$98.11
\]

\[
P_{33} = \frac{\$104.00}{1.06}
\]

\[
P_{33} = \$98.11
\]

\[
P_{34} = \frac{\$104.00/1.04}{1.06}
\]

\[
P_{34} = \$98.11
\]

\[
P_{35} = \frac{\$104.00}{1.06}
\]

\[
P_{35} = \$98.11
\]

\[
P_{36} = \frac{\$104.00/1.04}{1.06}
\]

\[
P_{36} = \$98.11
\]

\[
P_{37} = \frac{\$104.00}{1.06}
\]

\[
P_{37} = \$98.11
\]

\[
P_{38} = \frac{\$104.00/1.04}{1.06}
\]

\[
P_{38} = \$98.11
\]

\[
P_{39} = \frac{\$104.00}{1.06}
\]

\[
P_{39} = \$98.11
\]

\[
P_{40} = \frac{\$104.00/1.04}{1.06}
\]

\[
P_{40} = \$98.11
\]
Coupon bond price (cont)

- By risk-neutral pricing (cont)

\[ P_{0,0} = \frac{c + q_{0,0} \times P_{1,1} + (1 - q_{0,0}) \times P_{1,0}}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}} \]

\[ = \frac{\$4.00 + \frac{1}{2} \times \$97.23 + \frac{1}{2} \times \$99.07}{1.05} \]

\[ P_{0,0} = \$97.28 \]

\[ P_{1,0} = \$99.07 \]

\[ P_{1,1} = \$97.23 \]

\[ P_{2,0} = \frac{\$104.00}{1.04} = \$100 \]

\[ P_{2,1} = \frac{\$104.00}{1.05} = \$99.05 \]

\[ P_{2,2} = \frac{\$104.00}{1.06} = \$98.11 \]

\[ P_{3,0} = \$104.00 \]

\[ P_{3,1} = \$104.00 \]

\[ P_{3,2} = \$104.00 \]

\[ P_{3,3} = \$104.00 \]
European call on coupon bond

• 1-yr European style call option on 8% 1.5-yr coupon bond with strike price $K = 99.00$ pays max\[ 0, P_{2,?} - K \]

\[
\begin{align*}
V_{0,0} &= ? \\
V_{1,0} &= ? \\
V_{1,1} &= ? \\
V_{2,0} &= \text{max}\left[0, 100.00 - 99.00\right] = 1.00 \\
V_{2,1} &= \text{max}\left[0, 99.05 - 99.00\right] = 0.05 \\
V_{2,2} &= \text{max}\left[0, 98.11 - 99.00\right] = 0 \\
\end{align*}
\]
European call on coupon bond

- 1-yr European style call option on 8% 1.5-yr coupon bond with strike price $K = 99.00$ pays max[ 0, $P_{2,?} - K$ ]

\[ V_{0,0} = ? \]

\[ V_{1,0} = ? \]

\[ V_{1,1} = 0.0237 \]

\[ V_{2,0} = \text{max}[0, 100.00 - 99.00] = 1.00 \]

\[ V_{2,1} = \text{max}[0, 99.05 - 99.00] = 0.05 \]

\[ V_{2,2} = \text{max}[0, 98.11 - 99.00] = 0 \]

\[ V_{1,1} = \frac{q_{1,1} \times V_{2,2} + (1 - q_{1,1}) \times V_{2,1}}{(1 + r_{1,1}(1)/m)^{1 \times \Delta t \times m}} \]

\[ = \frac{0.5 \times 0.00 + 0.5 \times 0.05}{1.055} \]
European call on coupon bond

- 1-yr European style call option on 8% 1.5-yr coupon bond with strike price $K = $99.00 pays max[ 0, $P_{2,t} - K$ ]

\[
V_{0,0} = \text{?}
\]

- \[
V_{1,1} = \frac{q_{1,0} \times V_{2,1} + (1 - q_{1,0}) \times V_{2,0}}{(1 + r_{1,0}(1)/m)^{1 \times \Delta t \times m}}
\]

\[
= \frac{0.5 \times $0.05 + 0.5 \times $1.00}{1.045}
\]

- \[
V_{1,0} = $0.5024
\]

- \[
V_{2,2} = \max[0, $98.11 - $99.00]
\]

\[
= 0
\]

- \[
V_{2,1} = \max[0, $99.05 - $99.00]
\]

\[
= $0.05
\]

- \[
V_{2,0} = \max[0, $100.00 - $99.00]
\]

\[
= $1.00
\]
**European call on coupon bond**

- 1-yr European style call option on 8% 1.5-yr coupon bond with strike price $K = 99.00$ pays max[$0, P_{2,?} - K$]

V$_{2,2}$ = max[$0, 98.11 - 99.00$] = 0

V$_{2,1}$ = max[$0, 99.05 - 99.00$] = $0.05

V$_{2,0}$ = max[$0, 100.00 - 99.00$] = $1.00

V$_{0,0} = $0.2505

V$_{1,1} = $0.0237

V$_{1,0} = $0.5024

\[
V_{0,0} = \frac{q_{0,0} \times V_{1,1} + (1 - q_{0,0}) \times V_{1,0}}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}}
\]

\[
= \frac{0.5 \times $0.0237 + 0.5 \times $0.5024}{1.05}
\]
American put on coupon bond

- 1-yr American style put option on 8% 1.5-yr coupon bond with strike price $K = 99.00$ pays max[ 0, $K - P_i$ ]

\[
V_{2,2} = \max[0, 99.00 - 98.11] = 0.89
\]

\[
V_{2,1} = \max[0, 99.00 - 99.05] = 0.00
\]

\[
V_{2,0} = \max[0, 99.00 - 100.00] = 0.00
\]

\[
V_{1,1} = ?
\]

\[
V_{1,0} = ?
\]

\[
V_{0,0} = ?
\]
American put on coupon bond

- 1-yr American style put option on 8% 1.5-yr coupon bond with strike price $K = 99.00$ pays max[$0, K - P_i$]

\[ V_{0,0} = ? \]
\[ V_{1,0} = ? \]
\[ V_{1,1} = \frac{q_{1,1} \times V_{2,2} + (1 - q_{1,1}) \times V_{2,1}}{(1 + r_{1,1}(1)/m)^{1\times\Delta t\times m}} \times \text{exercise} \]
\[ = \max \left[ \frac{0.5 \times 0.89 + 0.5 \times 0.00}{1.055}, 99.00 - 97.23 \right] \]

\[ V_{2,2} = \max[0, 99.00 - 98.11] = 0.89 \]
\[ V_{2,1} = \max[0, 99.00 - 99.05] = 0.00 \]
\[ V_{2,0} = \max[0, 99.00 - 100.00] = 0.00 \]
American put on coupon bond

- 1-yr American style put option on 8% 1.5-yr coupon bond with strike price $K = $99.00 pays max[0, $K - P_i$]

\[
V_{0,0} = ?
\]

\[
V_{1,0} = $0.00
\]

\[
V_{1,1} = $1.7674
\]

\[
V_{2,0} = max[0,$99.00–$100.00] = $0.00
\]

\[
V_{2,1} = max[0,$99.00–$99.05] = $0.00
\]

\[
V_{2,2} = max[0,$99.00–$98.11] = $0.89
\]

\[
V_{1,0} = \max \left[ q_{1,0} \times V_{2,1} + (1 - q_{1,0}) \times V_{2,0}, \frac{K - P_{1,0}}{(1 + r_{1,0}/m)^1 \times \Delta t \times m} \right]
\]

\[
= \max \left[ 0.5 \times $0.00 + 0.5 \times $0.00, \frac{0.5 \times $99.00 - $99.07}{1.045} \right]
\]

\[
= \max [0,$0.00, $0.89]
\]

\[
= $0.89
\]
American put on coupon bond

- 1-yr American style put option on 8% 1.5-yr coupon bond with strike price $K = 99.00$ pays max[0, $K - P_i$]

\[ V_{2,2} = \max[0, 99.00 - 98.11] = 0.89 \]

\[ V_{2,1} = \max[0, 99.00 - 99.05] = 0.00 \]

\[ V_{2,0} = \max[0, 99.00 - 100.00] = 0.00 \]

\[ V_{1,1} = 1.7674 \]

\[ V_{1,0} = 0.00 \]

\[ V_{0,0} = 1.7150 \]

\[
V_{0,0} = \max \left[ \frac{q_{0,0} \times V_{1,1} + (1 - q_{0,0}) \times V_{1,0}}{(1 + r_{0,0}(1)/m)^{1 \times \Delta t \times m}}, K - P_{0,0} \right]_{\text{exercise}}
\]

\[
= \max \left[ \frac{0.5 \times 1.7674 + 0.5 \times 0.00}{1.05}, 99.00 - 97.28 \right]
\]
Callable Bond

• Suppose we want to price a 10% 5-yr coupon bond callable (by the issuer) at the end of year 3 at par
  – Step 1: Determine the price of the non-callable bond, $P_{NCB}$
  – Step 2: Determine the price of the call option on the non-callable bond with expiration after 3 years and strike price at par, $O_{NCB}$
  – Step 3: The price of the callable bond is
    \[ P_{CB} = P_{NCB} - O_{NCB} \]

• Intuition
  – The bondholder grants the issuer an option to buy back the bond
  – The value of this option must be subtracted from the price the bondholder pays the issuer for the non-callable bond
Putable Bond

- Suppose we want to price a 10% 5-yr coupon bond putable (by the bondholder to the issuer) at the end of year 3 at par
  - Step 1: Determine the price of the non-putable bond, $P_{NPB}$
  - Step 2: Determine the price of the put option on the non-putable bond with expiration after 3 years and strike price at par, $O_{NPB}$
  - Step 3: The price of the putable bond is
    \[ P_{PB} = P_{NPB} + O_{NPB} \]

- Intuition
  - The bond issuer grants the holder an option to sell back the bond
  - The value of this option must be added to the price the bondholder pays the issuer for the non-putable bond
Spot rate process

- Binomial trees are based on spot rate values $r_{i,j}(1)$ and risk-neutral probabilities $q_{i,j}$.
- In single-factor models, these values are determined by a risk-neutral spot rate process of the form

$$r_{t+\Delta t}(1) - r_t(1) = \mu \left[ r_t(1), t \right] \times \Delta t + \sigma \left[ r_t(1), t \right] \times \sqrt{\Delta t} \times \epsilon_t$$

where

- Mean[$\epsilon_t$] = 0
- Var[$\epsilon_t$] = 1

such that

- Mean[$r_{t+\Delta t}(1) - r_t(1)$] = $\mu \left[ r_t(1), t \right] \times \Delta t$
- Var[$r_{t+\Delta t}(1) - r_t(1)$] = $\sigma \left[ r_t(1), t \right]^2 \times \Delta t$
Spot rate process (cont)

- In an $N$-factor models, these values are determined by a risk-neutral spot rate process of the form

$$r_t(1) = z_{1,t} + z_{2,t} + \cdots + z_{N,t}$$

with

$$z_{i,t + \Delta t} - z_{i,t} = \mu_i \left[ z_{1,t}, z_{2,t}, \cdots, z_{N,t}, t \right] \times \Delta t +$$

$$\sigma_i \left[ z_{1,t}, z_{2,t}, \cdots, z_{N,t}, t \right] \times \sqrt{\Delta t} \times \epsilon_{1,t}$$

and

$$\text{Mean}[\epsilon_{i,t}] = 0 \quad \text{Var}[\epsilon_{i,t}] = 1$$
**Drift function**

- **Case 1: Constant drift**

\[
 r_{t+\Delta t} - r_t = \lambda \times \Delta t + \sigma \times \sqrt{t} \times \epsilon_t
\]

with

\[
 \epsilon_t \sim \mathcal{N}[0, 1]
\]

- **Implied distribution of 1-period spot rates**

\[
 r_{t+\Delta t} \sim \mathcal{N}\left[ \frac{r_t + \lambda \times \Delta t}{\text{mean}}, \frac{\sigma^2 \times \Delta t}{\text{var}} \right]
\]
Drift function (cont)

- Binomial tree representation

\[
\begin{align*}
q = 1/2 & \quad r_{0,0} + \lambda \times \Delta t + \sigma \times v \Delta t \\
q = 1/2 & \quad r_{0,0} + 2 \times \lambda \times \Delta t + 2 \times \sigma \times v \Delta t \\
r_{0,0} & \quad r_{0,0} + \lambda \times \Delta t - \sigma \times v \Delta t \\
r_{0,0} & \quad r_{0,0} + 2 \times \lambda \times \Delta t - 2 \times \sigma \times v \Delta t
\end{align*}
\]

- Properties
  - No mean reversion
  - No heteroskedasticity
  - Spot rates can become negative, but not if we model ln[r(1)]
    \[ \Rightarrow \text{“Rendleman-Bartter model”} \]
  - 2 parameters
  - \[ \Rightarrow \text{fit only 2 spot rates} \]
Drift function (cont)

- Example
  - $r_{0,0} = 5\%$
  - $\lambda = 1\%$
  - $\sigma = 2.5\%$
  - $\Delta t = 1/m$ with $m = 2$

\[
\begin{align*}
q &= \frac{1}{2} \\
r_{0,0} &= 5\% \\
r_{1,0} &= 3.73\% \\
r_{1,1} &= 7.27\% \\
r_{2,0} &= 2.47\% \\
r_{2,1} &= 6.00\% \\
r_{2,2} &= 9.54\%
\end{align*}
\]
**Drift function (cont)**

- **Case 2: Time-dependent drift**
  \[
  r_{t+\Delta t} - r_t = \underbrace{\lambda(t) \times \Delta t}_{\text{drift fct}} + \sigma \times \sqrt{t} \times \epsilon_t
  \]
  with
  \[
  \epsilon_t \sim N[0, 1]
  \]

- **Implied distribution of 1-period spot rates**
  \[
  r_{t+\Delta t} \sim N\left[r_t + \lambda(t) \times \Delta t, \sigma^2 \times \Delta t \right]
  \]

- **Ho and Lee (1986, *J. of Finance*) ⇒ “Ho-Lee model”**
Drift function (cont)

• Binomial tree representation

\[
\begin{align*}
q = 1/2 & \quad r_{0,0} + \lambda(1) \times \Delta t + \sigma \times \nu \times \Delta t \\
q = 1/2 & \quad r_{0,0} + \lambda(1) + \lambda(2) \times \Delta t + 2 \times \sigma \times \nu \times \Delta t
\end{align*}
\]

• Properties
  – No heteroskedasticity
  – Spot rates can become negative, but not if we model \( \ln[r(1)] \)
    \( \Rightarrow \) “Salomon Brothers model”
  – Arbitrarily many parameters
    \( \Rightarrow \) fit term structure of spot rates but not necessarily spot rate volatilities (i.e., derivative prices)
Drift function (cont)

• Case 3: Mean reversion

\[ r_{t+\Delta t} - r_t = \kappa \times \left[ \theta - r_t(1) \right] \times \Delta t + \sigma \times \sqrt{t} \times \epsilon_t \]

with

\[ \epsilon_t \sim N[0, 1] \]

• Implied distribution of 1-period spot rates

\[ r_{t+\Delta t} \sim N \left[ r_t + \kappa \times \left[ \theta - r_t(1) \right] \times \Delta t, \sigma^2 \times \Delta t \right] \]

Drift function (cont)

• Binomial tree representation

\[
q = \frac{1}{2} \\
\begin{align*}
r_{0,0} + \kappa \times (\theta - r_{0,0}) \times \Delta t + \sigma \times \nu \times \Delta t & \\
r_{1,0} + \kappa \times (\theta - r_{1,0}) \times \Delta t - \sigma \times \nu \times \Delta t & \\
r_{1,1} + \kappa \times (\theta - r_{1,1}) \times \Delta t + \sigma \times \nu \times \Delta t & \\
r_{1,0} + \kappa \times (\theta - r_{1,0}) \times \Delta t - \sigma \times \nu \times \Delta t & \\
\end{align*}
\]

• Properties
  - Non-recombining, but can be fixed
  - No heteroskedasticity
  - Spot rates can become negative, but not if we model \(\ln[r(1)]\)
  - 3 parameters
    \(\Rightarrow\) fit only 3 spot rates
Drift function (cont)

- Example
  - $r_{0,0} = 5\%$
  - $\theta = 10\%$
  - $\kappa = 0.25$
  - $\sigma = 2.5\%$
  - $\Delta t = 1/m$ with $m = 2$

With $\kappa = 0$

$$r_{2,2} = 8.54\%$$

$$r_{2,2} = 9.49\%$$

$$r_{2,1} = 5.95\%$$

$$r_{2,1} = 5.00\%$$

$$r_{2,0} = 2.86\%$$

$$r_{2,0} = 1.46\%$$
Drift function (cont)

- Example
  - \( r_{0,0} = 15\% \)
  - \( \theta = 10\% \)
  - \( \kappa = 0.25 \)
  - \( \sigma = 2.5\% \)
  - \( \Delta t = 1/m \) with \( m = 2 \)

\[
egin{align*}
q &= 1/2 & r_{2,2} &= 17.14\% \\
q &= 1/2 & r_{1,2} &= 16.77\% \\
r_{0,0} &= 15\% & r_{2,1} &= 13.61\% \\
r_{1,0} &= 12.61\% & r_{2,1} &= 14.05\% \\
r_{2,0} &= 10.51\% & r_{2,0} &= 11.46\% \\
\end{align*}
\]

With \( \kappa = 0 \)

\[
egin{align*}
r_{2,2} &= 18.54\% \\
r_{2,1} &= 15.00\% \\
r_{2,0} &= 11.46\% \\
\end{align*}
\]
Volatility function

- **Case 1: Square-root volatility**

\[ r_{t+\Delta t} - r_t = \lambda \times \Delta t + \sigma \times \sqrt{r_t} \times \sqrt{\Delta t} \times \epsilon_t \]

with

\[ \epsilon_t \sim N[0, 1] \]

- **Implied distribution of 1-period spot rates**

\[ r_{t+\Delta t} \sim N \left[ r_t + \lambda \times \Delta t, \sigma^2 \times r_t \times \Delta t \right] \]

- Cox, Ingersoll, and Ross (1985, *Econometrics*) \( \Rightarrow \) “CIR model”
Volatility function (cont)

• Binomial tree representation

\[ q = 1/2 \]
\[ r_{0,0} + \lambda \times \Delta t + \sigma \times v \times r_{0,0} \times v \times \Delta t \]

\[ q = 1/2 \]
\[ r_{1,1} + \lambda \times \Delta t + \sigma \times v \times r_{1,1} \times v \times \Delta t \]
\[ r_{1,0} + \lambda \times \Delta t + \sigma \times v \times r_{1,0} \times v \times \Delta t \]
\[ r_{0,0} + \lambda \times \Delta t - \sigma \times v \times r_{0,0} \times v \times \Delta t \]

• Properties

– Non-recombining, but can be fixed
– No mean-reversion, but can be fixed by using different drift function
– Spot rates can become negative, but not as \( \Delta t \to 0 \)
– 1 volatility parameter (and arbitrarily many drift parameters)
  \( \Rightarrow \) fit term structures of spot rates but only 1 spot rate volatility
Volatility function (cont)

- Example
  - $r_{0,0} = 5\%$
  - $\lambda = 1\%$
  - $\sigma = 11.18\% \Rightarrow \sigma \times \nu r_{0,0} = 2.5\%$
  - $\Delta t = 1/m$ with $m = 2$

<table>
<thead>
<tr>
<th>$r_{2,2}$</th>
<th>$r_{2,1}$</th>
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<tr>
<td>9.90%</td>
<td>5.64%</td>
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<td>9.54%</td>
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With constant volatility
Volatility function (cont)

- **Case 2: Time-Dependent volatility**

\[ r_{t+\Delta t} - r_t = \lambda \times \Delta t + \sigma(t) \times \sqrt{\Delta t} \times \epsilon_t \]

with

\[ \epsilon_t \sim N[0, 1] \]

- **Implied distribution of 1-period spot rates**

\[ r_{t+\Delta t} \sim N\left( r_t + \lambda \times \Delta t, \sigma(t)^2 \times \Delta t \right) \]

- Hull and White (1993, *J. of Financial and Quantitative Analysis*)
  \( \Rightarrow \) “Hull-White model”
Volatility function (cont)

- Binomial tree representation
  \[
  \begin{align*}
  q = 1/2 & \quad r_{0,0} + \lambda \times \Delta t + \sigma(1) \times \nu \Delta t \\
  r_{0,0} & \quad r_{0,0} + \lambda \times \Delta t - \sigma(1) \times \nu \Delta t
  \end{align*}
  \]

- Properties
  - Non-recombining, but can be fixed
  - No mean-reversion, but can be fixed by using different drift function
  - Spot rates can become negative, but not if we model \(\ln[r(1)]\)
    \(\Rightarrow\) “Black-Karasinski model” and “Black-Derman-Toy model”
  - Arbitrarily many volatility and drift parameters
    \(\Rightarrow\) fit term structures of spot rates and volatilities
Calibration

• To calibrate parameters of a factor model to bonds prices
  – Step 1: Pick arbitrary parameter values
  – Step 2: Calculate implied 1-period spot rate tree
  – Step 3: Calculate model prices for liquid securities
  – Step 4: Calculate model pricing errors given market prices
  – Step 5: Use solver to find parameter values which minimize the sum of squared pricing errors
Constant drift example

• Step 1: Pick arbitrary parameter values

Parameters

? 0.00%
s 0.10%

Observed 1-period spot rate

r(1) 6.21%
Constant drift example (cont)

• Step 2: Calculate implied 1-period spot rate tree

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Constant drift example (cont)

• Step 3: Calculate model prices for liquid securities
  – E.g., for a 2.5-yr STRIPS

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<th>$ 96.86</th>
<th>$ 93.87</th>
<th>$ 91.05</th>
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Factor models
### Constant drift example (cont)

- Step 3: Calculate model prices for liquid securities (cont)

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<tr>
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<th>Model implied periods</th>
<th>Model implied spot rate</th>
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</thead>
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<td>? 0.00%</td>
<td>0.5</td>
<td>6.21%</td>
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<tr>
<td>s 0.10%</td>
<td>1.0</td>
<td>6.21%</td>
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<td>6.21%</td>
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**Observed 1-period spot rate**

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<thead>
<tr>
<th>Periods</th>
<th>Model implied spot rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.5</td>
<td>6.21%</td>
</tr>
<tr>
<td>3.0</td>
<td>6.21%</td>
</tr>
</tbody>
</table>

| r(1) 6.21% | 3.5 | 6.21% |
|           | 4.0 | 6.21% |
|           | 4.5 | 6.21% |
|           | 5.0 | 6.21% |
### Constant drift example (cont)

- **Step 4: Use solver**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model implied</th>
<th>Observed</th>
<th>Pricing</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Periods</td>
<td>spot rate</td>
<td>spot rate</td>
</tr>
<tr>
<td>?</td>
<td>0.5</td>
<td>6.21%</td>
<td>6.21%</td>
</tr>
<tr>
<td>s</td>
<td>1.0</td>
<td>6.21%</td>
<td>6.41%</td>
</tr>
<tr>
<td></td>
<td>1.5</td>
<td>6.21%</td>
<td>6.48%</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>6.21%</td>
<td>6.56%</td>
</tr>
<tr>
<td>Observed 1-period spot rate</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.5</td>
<td>6.21%</td>
<td>6.62%</td>
</tr>
<tr>
<td>r(1)</td>
<td>3.0</td>
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<td>6.71%</td>
</tr>
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<td>6.21%</td>
<td>6.80%</td>
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<tr>
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<td>4.0</td>
<td>6.21%</td>
<td>6.87%</td>
</tr>
<tr>
<td></td>
<td>4.5</td>
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</tr>
<tr>
<td></td>
<td>5.0</td>
<td>6.21%</td>
<td>6.97%</td>
</tr>
</tbody>
</table>

Sum of squared errors: 0.0002521

Minimize sum of squared errors by choice of parameters
### Constant drift example (cont)

#### Solution

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Model implied</th>
<th>Observed</th>
<th>Pricing error</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Periods</td>
<td>spot rate</td>
<td>spot rate</td>
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<td>$? 0.57%$</td>
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<td>6.21%</td>
</tr>
<tr>
<td>$s 3.63%$</td>
<td>1.0</td>
<td>6.34%</td>
<td>6.41%</td>
</tr>
<tr>
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<td>1.5</td>
<td>6.45%</td>
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<tr>
<td></td>
<td>2.0</td>
<td>6.56%</td>
<td>6.56%</td>
</tr>
<tr>
<td><strong>Observed 1-period spot rate</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>2.5</td>
<td>6.65%</td>
<td>6.62%</td>
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<tr>
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<td>3.0</td>
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<td>6.71%</td>
</tr>
<tr>
<td>$r(1) 6.21%$</td>
<td>3.5</td>
<td>6.81%</td>
<td>6.80%</td>
</tr>
<tr>
<td></td>
<td>4.0</td>
<td>6.87%</td>
<td>6.87%</td>
</tr>
<tr>
<td></td>
<td>4.5</td>
<td>6.92%</td>
<td>6.92%</td>
</tr>
<tr>
<td></td>
<td>5.0</td>
<td>6.96%</td>
<td>6.97%</td>
</tr>
</tbody>
</table>

Sum of squared errors: $7.882E-07$