1 Convex optimization over the simplex constraint

We consider the following optimization problem over the simplex:

\[
\begin{align*}
\text{Minimize} & \quad f(x) \\
\text{Subject To} & \quad e^T x = n; \quad x \geq 0,
\end{align*}
\]

where \(e\) is the vector of all ones. This problem is to minimize a nonlinear function with a Simplex constraint. Such a problem is considered in [7], where function \(f(x)\) does not need to be convex and a FPTAS algorithm was developed for computing an approximate KKT point of general quadratic programming. The following algorithm and analysis resemble those in [7].

We assume that \(f(x)\) is a convex function in \(x \in R^n\) and \(f(x^*) = 0\) where \(x^*\) is a minimizer of the problem. Furthermore, we make a standard Lipschitz assumption such that

\[
f(x + d) - f(x) \leq \nabla f(x)^T d + \frac{\gamma}{2} \|d\|^2,
\]

where positive \(\gamma\) is the Lipschitz parameter. Note that any homogeneous linear feasibility problem, e.g., the canonical Karmarkar form in [2]:

\[
Ax = 0; \\
e^T x = n; \\
x \geq 0.
\]

can be formulated as the model with \(f(x) = \frac{1}{2} \|Ax\|^2\) and \(\gamma\) as the half of the largest eigenvalue of matrix \(A^T A\).

Furthermore, any linear programming problem in the standard form and its dual

\[
\begin{align*}
& \text{Minimize} \quad c^T x \\
& \text{Subject to} \quad Ax = b; \quad x \geq 0;
\end{align*}
\]

\[
\begin{align*}
& \text{Maximize} \quad b^T y \\
& \text{Subject to} \quad A^T y + s = c; \quad s \geq 0
\end{align*}
\]
can be represented as a homogeneous linear feasibility problem (Ye et al. [5]):

\[ A x - b \tau = 0; \]
\[ -A^T y - s + c \tau = 0; \]
\[ b^T y - c^T x - \kappa = 0; \]
\[ e^T x + e^T s + \tau + \kappa = 2n + 2; \]
\[ (x, s, \tau, \kappa) \geq 0. \]

We consider the potential function (e.g., see [2, 4, 1, 6])

\[ \phi(x) = \rho \ln(f(x)) - \sum_j \ln(x_j), \]

(alternatively, one may consider barrier function \( b_\mu(x) = f(x) - \mu \sum_j \ln(x_j) \) for a small fixed \( \mu \))

where \( \rho \geq n \) over the simplex. Clearly, if we start from \( x^0 = e \), the analytic center of the simplex, and generate a sequence of points \( x^k, k = 1, ..., \) whose potential value is strictly decreased, then when

\[ \phi(x^k) - \phi(x^0) \leq -\rho \ln(1/\epsilon), \]

we must have

\[ \rho \ln(f(x^k)) - \rho \ln(f(x^0)) \leq -\rho \ln(1/\epsilon) \]

or

\[ \frac{f(x^k)}{f(x^0)} \leq \epsilon. \]

This is because on the simplex

\[ \sum_j \ln(x_j^k) \leq \sum_j \ln(x_j^0), \forall k = 1, ... . \]

We now describe a first order steepest descent potential reduction algorithm in the next section.

## 2 Steepest-Descent Potential Reduction and Complexity Analysis

Note that the gradient vector of the potential function of \( x > 0 \) is

\[ \nabla \phi(x) = \frac{\rho}{f(x)} \nabla f(x) - X^{-1} e. \]

where in this note \( X \) denotes the diagonal matrix whose diagonal entries are elements of vector \( x \).

The following lemma is well known in the literature of interior-point algorithms ([2, 1, 6]):

**Lemma 1.** Let \( x > 0 \) and \( \|X^{-1}d\| \leq \beta < 1 \). Then

\[ -\sum_j \ln(x_j + d_j) + \sum_j \ln(x_j) \leq -e^T X^{-1} d + \frac{\beta^2}{2(1 - \beta)}. \]

\[ 2 \]
Lemma 2. For any \( x > 0 \) and \( x \neq x^* \), a matrix \( A \in \mathbb{R}^{m \times n} \) with \( Ax = Ax^* \), and a vector \( \bar{\lambda} \in \mathbb{R}^m \), consider vector
\[
p(x) = X \left( \nabla \phi(x) - A^T \bar{\lambda} \right).
\]
Then,
\[
\|p(x)\| \geq 1.
\]

Proof. First,
\[
p(x) = X \left( \frac{\rho}{f(x)} \nabla f(x) - X^{-1} e - A^T \bar{\lambda} \right) = \frac{\rho}{f(x)} X \left( \nabla f(x) - \frac{f(x)}{\rho} A^T \bar{\lambda} \right) - e.
\]
If any entry of \( (\nabla f(x) - \frac{f(x)}{\rho} A^T \bar{\lambda}) \) is equal or less than 0, then \( \|p(x)\| \geq \|p(x)\|_{\infty} \geq 1 \). On the other hand, if \( (\nabla f(x) - \frac{f(x)}{\rho} A^T \bar{\lambda}) > 0 \), we have \( (\nabla f(x) - \frac{f(x)}{\rho} A^T \bar{\lambda})^T x^* \geq 0 \). Then, from convexity and \( Ax = Ax^* \),
\[
f(x^*) - f(x) \geq \nabla f(x)^T (x^* - x) = \left( \nabla f(x) - \frac{f(x)}{\rho} A^T \bar{\lambda} \right)^T (x^* - x).
\]
Thus, from \( f(x^*) = 0 \)
\[
f(x) \leq \left( \nabla f(x) - \frac{f(x)}{\rho} A^T \bar{\lambda} \right)^T x.
\]
Furthermore,
\[
\|p(x)\|^2 = \frac{\rho^2}{f(x)^2} \|X \left( \nabla f(x) - \frac{f(x)}{\rho} A^T \bar{\lambda} \right) \|^2 - 2 \frac{\rho^2}{f(x)} \left( \nabla f(x) - \frac{f(x)}{\rho} A^T \bar{\lambda} \right)^T x + n
\geq \frac{\rho^2}{n f(x)^2} \|X \left( \nabla f(x) - \frac{f(x)}{\rho} A^T \bar{\lambda} \right) \|^2 - 2 \frac{\rho}{f(x)} \left( \nabla f(x) - \frac{f(x)}{\rho} A^T \bar{\lambda} \right)^T x + n
\geq \frac{\rho^2}{n} \left( \frac{\nabla f(x) - \frac{f(x)}{\rho} A^T \bar{\lambda}}{f(x)} \right)^2 - 2 \rho \left( \frac{\nabla f(x) - \frac{f(x)}{\rho} A^T \bar{\lambda}}{f(x)} \right)^T x + n
= \frac{(\rho z)^2}{n} - 2 \rho z + n = \frac{1}{n} (\rho z - n)^2,
\]
where
\[
z = \frac{\nabla f(x) - \frac{f(x)}{\rho} A^T \bar{\lambda}}{f(x)} \geq 1.
\]
The above quadratic function of \( z \) has the minimizer at \( z = 1 \) if \( \rho \geq n \), so that
\[
\frac{1}{n} (\rho z - n)^2 \geq \frac{1}{n} (\rho - n)^2 \geq 1
\]
for \( \rho \geq n + \sqrt{n} \).

For any given \( x > 0 \) in the simplex and any \( d \) with \( e^T d = 0 \),
\[
f(x + d) - f(x) \leq \nabla f(x)^T d + \frac{\gamma}{2} ||d||^2 \leq \nabla f(x)^T d + \frac{\gamma}{2} \|X^{-1} X d\|^2 \leq \nabla f(x)^T d + \frac{\gamma}{2} \|X^{-1} d\|^2,
\]

3
where the last inequality is due to $\|X\| \leq 1$. Let $\|X^{-1}d\| = \beta < 1$ and $x^+ = x + d = X(e + X^{-1}d) > 0$. Then, from Lemma 1
\[
\phi(x^+) - \phi(x) \leq \rho \ln \left( 1 + \frac{\nabla f(x)^T \frac{d + \frac{1}{2} \|X^{-1}d\|^2}{f(x)}}{f(x)} \right) - e^T X^{-1}d + \frac{\beta^2}{2(1-\beta)}
\]
\[
= \nabla \phi(x)^T d + \frac{\rho^2 f(x) \beta}{2f(x)} \beta^2 + \frac{\beta^2}{2(1-\beta)}.
\]
The first order steepest descent potential reduction algorithm would update $x$ by solving

\begin{align}
\text{Minimize} &\quad \nabla \phi(x)^T d \\
\text{Subject to} &\quad e^T d = 0, \|X^{-1}d\| \leq \beta;
\end{align}

or

\begin{align}
\text{Minimize} &\quad \nabla \phi(x)^T Xd' \\
\text{Subject to} &\quad e^T Xd' = 0, \|d'\| \leq \beta;
\end{align}

where parameter $\beta < 1$ is yet to be determined.

Let the scaled gradient projection vector
\[
p(x) = \left( I - \frac{1}{\|x\|^2} X e e^T X \right) X \nabla \phi(x) = X \left( \frac{\rho}{f(x)} (\nabla f(x) - e \cdot \lambda(x)) \right) - e,
\]
where
\[
\lambda(x) = \frac{e^T X^2 \nabla \phi(x) \cdot f(x)}{\|x\|^2 \cdot \rho}.
\]
Then the minimizer of problem (2) would be
\[
d = -\frac{\beta}{\|p(x)\|} X p(x),
\]
and
\[
\nabla \phi(x)^T d = -\frac{\beta}{\|p(x)\|} \|p(x)\|^2 = -\beta \|p(x)\| \leq -\beta,
\]
since $\|p(x)\| \geq 1$ based on Lemma 2.

Thus,
\[
\phi(x^+) - \phi(x) \leq -\beta + \frac{\rho \gamma}{2f(x)} \beta^2 + \frac{\beta^2}{2(1-\beta)}
\]
For $\beta \leq 1/2$, the above quantity is less than
\[
-\beta + \left( 2 + \frac{\rho \gamma}{f(x)} \right) \beta^2 / 2.
\]
Thus, one can choose $\beta$ to minimize the quantity at
\[
\beta = \frac{1}{2 + \frac{\rho \gamma}{f(x)}} \leq 1/2
\]
so that
\[
\phi(x^+) - \phi(x) \leq \frac{-f(x)}{2(f(x) + 2\rho \gamma)}.
\]
One can see that the larger value of $f(x)$, the greater reduction of the potential function.

Starting from $x^0 = \frac{1}{n}e$, we iteratively generate $x^k$, $k = 1, \ldots$, such that

$$\phi(x^{k+1}) - \phi(x^k) \leq \frac{-f(x^k)}{2(f(x^k) + 2\rho\gamma)} \leq \frac{-f(x^k)}{2(f(x^0) + 2\rho\gamma)} \leq \frac{-f(x^k)}{4\max\{f(x^0), 2\rho\gamma\}}.$$  

The second inequality is due to $f(x^k) < f(x^0)$ from $\phi(x^k) < \phi(x^0)$ for all $k \geq 1$ and $x^0$ is the analytic center of the simplex.

Thus, if $\frac{f(x^k)}{f(x^0)} \geq \epsilon$ for $1 \leq k \leq K$, we must have

$$\phi(x^0) - \phi(x^K) \leq \rho \ln\left(\frac{1}{\epsilon}\right),$$

so that

$$\sum_{k=1}^{K} \frac{f(x^k)}{4\max\{f(x^0), 2\rho\gamma\}} \leq \rho \ln\left(\frac{1}{\epsilon}\right)$$

or

$$K\epsilon f(x^0) \leq 4\max\{f(x^0), 2\rho\gamma\} \rho \ln\left(\frac{1}{\epsilon}\right).$$

Note that $\rho = n + \sqrt{n} \leq 2n$. We conclude

**Theorem 3.** The steepest descent potential reduction algorithm generates a $x^k$ with $f(x^k)/f(x^0) \leq \epsilon$ in no more than

$$4(n + \sqrt{n}) \max\{1, 2(n + \sqrt{n})\gamma/f(x^0)\} \epsilon \ln\left(\frac{1}{\epsilon}\right)$$

steps.

### 3 Extension, Implementation and Possible Further Analysis

**Question 1:** Develop a similar analysis for solving

\[
\begin{align*}
\text{Minimize} & \quad f(x) \\
\text{Subject To} & \quad 0 \leq x_j \leq 2, \; \forall j = 1, \ldots, n,
\end{align*}
\]

where we start $x^0 = e$, the analytic center of the BOX constraint.

**Question 2:** Implement the algorithm and perform numerical tests to solve for

$$f(x) = \frac{1}{2} \|Ax\|^2$$

either in (1), or (3), or both.

**Question 3:** Implement the algorithm and perform numerical tests to solve for

$$f(x) = \frac{1}{2} \|(AA^T)^{-1/2}Ax\|^2,$$
and compare the performance with that in Question 2. This can be viewed as one-time preconditioning.

**Question 4:** Test your implementation on homogeneous and self LP models for various linear programs (feasible or infeasible), where you may eliminate free variables $y$ from the formulation.

### 4 Extension to MDP

Consider the MDP problem

$$\text{maximize}_y \sum_{i=1}^{m} y_i$$

subject to

$$y_i - \gamma p^T_j y \leq c_j, \ j \in A_i$$

$$\vdots$$

$$y_i - \gamma p^T_j y \leq c_j, \ j \in A_i$$

$$\vdots$$

$$y_m - \gamma p^T_j y \leq c_j, \ j \in A_m.$$ 

One can construct a potential/barrier function for a small fixed $\mu$ as

$$b_\mu(y) = -e^T y - \mu \sum_j \log(c_j - y_i + \gamma p^T_j y),$$

or

$$\psi(y) = \rho \log(z - e^T y) - \sum_j \log(c_j - y_i + \gamma p^T_j y)$$

where $\rho \geq n$ and $z$ is a upper bound on the maximal value of the MDP problem.

**Question 5:** The problem becomes a unconstrained problem when start $y^0 = -\Delta e$ (in the interior of the feasible region) for a big enough $\Delta$. You may apply the (stochastic) steepest descent method, the conjugate gradient method, the BFGS method, or any deep-learning method, etc, and do numerical experiments. The stochastic gradient would be sample some log terms in the summation and sum up their gradient vectors.

### References


