Pricing Parimutuel Digital Call Auction (Bonus Case Study)

Goldman Sachs and Deutsche Bank plan to jointly launch the first of a series of options on economic data in the first week of October. They are jointly marketing the products, which are based on the Parimutuel Digital Call Auction (PDCA) technology developed by US financial technology firm Longitude. The first auction will be on the September US non-farm payroll figure. If successful, the products would be the first tradable instruments for hedging exposures to, or taking positions against, economic indicators such as employment, inflation and GDP.

Parimutuel principles are widely used as an alternative to fixed odds gambling in which a bookmaker acts as a dealer by quoting fixed rates of return on specified wagers. A parimutuel game is conducted as a call auction in which odds are allowed to fluctuate during the betting period until the betting period is closed or the auction "called." The prices or odds of wagers are set based upon the relative amounts wagered on each risky outcome.

The following is a mathematical description of the problem. First, there are m risk states that are mutually exclusive and exactly one of them will be true at the maturity. A contract on a state is a paper agreement so that on maturity it is worth a pre-determined notional $\$w$ if it is on the winning state and worth \$0 if is not on the winning state. There are n orders betting on one or a combination of states and each order has a price limit and a quantity limit. For example, the jth order is given as $(\mathbf{a}_j \in R^m_+,\ \pi_j \in R_+,\ q_j \in R_+)$, where $\mathbf{a}_j$ is the combination betting vector where each component is either 1 or 0

$$\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \dots \\ a_{mj} \end{pmatrix},$$

where 1 is winning state bet and 0 is non-winning state bet; π_j is the price limit for one such a contract, and q_j is the maximum number of contracts the better like to have.

Let x_j, $j = 1, ..., n$, be the number of contracts awarded to the jth better, and the Auction Settling Price (ASP) of each state be p_i, $i = 1, ..., m$. Then, the jth better will pay the amount of $\mathbf{p}^T \mathbf{a}_j \cdot x_j$ and the total amount paid to the auction organizer is

$$\sum_{j=1}^{n} \mathbf{p}^T a_j \cdot x_j = \mathbf{p}^T A \mathbf{x}$$

where

$$A = (a_1,\ a_2,\ ...,\ a_n)\,.$$

If at the end the ith state is the winning state, then the auction organizer need to pay back

$$w \cdot \left(\sum_{j=1}^{n} a_{ij} x_j \right).$$

The first question is how to decide $\mathbf{x} \in R^n$ or find the qualified order and decide the award quantities; while the second question is how to decide the ASP $\mathbf{p} \in R^m$.

A linear programming model was proposed to decide which order is qualified or not (Ng and Yan, Department of Mathematics, The Hong Kong University, 2004):

$$\begin{array}{rrl} \max & \pi^T \mathbf{x} - w \cdot s & \\ \text{s.t.} & A\mathbf{x} - \mathbf{e} \cdot s & \leq 0, \\ & \mathbf{x} & \leq \mathbf{q}, \\ & \mathbf{x} & \geq 0. \end{array}$$

One can see that $\pi^T \mathbf{x}$ represents the potentially amount money can be collected by the auction organizer and $w \cdot s$ is the largest possible amount need to pay back to the betters. One method is to solve the linear program to obtain an optimal solution $\mathbf{x}^*$. Then to find $\mathbf{p}$ such that it satisfies the following "fairness" conditions:

$x_j^* = q_j$	$\mathbf{a}_j^T \mathbf{p} \leq \pi_j$
$0 < x_j^* < q_j$	$\mathbf{a}_j^T \mathbf{p} = \pi_j$
$x_j = 0$	$\mathbf{a}_j^T \mathbf{p} \geq \pi_j$

In this case project, you would find "better" or "newer" pricing models for solving this problem. It is part of your job to determine one or more criteria for "goodness". Here I list several topics you may work on

- Do literature search and understand the state of art models and technologies in solving the problem. I have attached few. You may also go to http://www.gs.com/econderivs/ to learn more of economic derivatives.
- Consider the LP model presented earlier. Its dual problem is

$$\begin{array}{rrl} \min & \mathbf{q}^T \mathbf{y} & \\ \text{s.t.} & A^T \mathbf{p} + \mathbf{y} & \geq \pi, \\ & \mathbf{e}^T \mathbf{p} & = w, \\ & (\mathbf{p}, \mathbf{y}) & \geq 0, \end{array}$$

  where $\mathbf{e}$ is the vector of all ones. Write down the complementarity condition of the primal and dual, and show the optimal dual solution $\mathbf{p}^*$ must satisfy the "fairness" conditions.
- The number of orders may be large and they come in sequentially. Is there a dynamic pricing method to deal with the problem? That is, pricing the problem based on current orders; then update the price (if necessary) when new orders come.
- Suppose information, such as the probability distribution on which state to win, is available for the auction organizer. How should this information be incorporated into the model? How does the model to handle risk?
- Implement your model and conduct some simulation experiments.