Mathematical Preliminaries

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Chapters 1 and Appendixes A,B.1-B.2,C.1
Real $n$-Space; Euclidean Space

- $\mathcal{R}$, $\mathcal{R}_+$, int $\mathcal{R}_+$
- $\mathbb{R}^n$, $\mathbb{R}_+^n$, int $\mathbb{R}_+^n$
- $x \geq y$ means $x_j \geq y_j$ for $j = 1, 2, \ldots, n$
- $\mathbf{0}$: all zero vector; and $\mathbf{e}$: all one vector
- Inner-Product:
  \[ x \cdot y := x^T y = \sum_{j=1}^{n} x_j y_j \]
- Norm: $\|x\|_2 = \sqrt{x^T x}$, $\|x\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$, $\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}$
- The dual of the $p$ norm, denoted by $\|\cdot\|^*$, is the $q$ norm, where $\frac{1}{p} + \frac{1}{q} = 1$
- Column vector:
  \[ x = (x_1; x_2; \ldots; x_n) \]
and row vector:

\[ \mathbf{x} = (x_1, x_2, \ldots, x_n) \]

- A set of vectors \( \mathbf{a}_1, \ldots, \mathbf{a}_m \) is said to be **linearly dependent** if there are scalars \( \lambda_1, \ldots, \lambda_m \), not all zero, such that the linear combination

\[ \sum_{i=1}^{m} \lambda_i \mathbf{a}_i = 0 \]

- A linearly independent set of vectors that span \( \mathbb{R}^n \) is a **basis**.

- For a sequence \( \mathbf{x}^k \in \mathbb{R}^n, k = 0, 1, \ldots \), we say it is a **contraction** sequence if there is an \( \mathbf{x}^* \in \mathbb{R}^n \) and a scalar constant \( 0 < \gamma < 1 \) such that

\[ \| \mathbf{x}^{k+1} - \mathbf{x}^* \| \leq \gamma \| \mathbf{x}^k - \mathbf{x}^* \|, \quad \forall k \geq 0. \]
Matrices

- $A \in \mathbb{R}^{m \times n}$; $a_i$, the $i$th row vector; $a_{j\cdot}$, the $j$th column vector; $a_{ij}$, the $i,j$th entry
- $0$: all zero matrix, and $I$: the identity matrix
- The null space $\mathcal{N}(A)$, the row space $\mathcal{R}(A^T)$, and they are orthogonal.
- $\det(A)$, $\text{tr}(A)$: the sum of the diagonal entries of $A$
- Inner Product:
  \[ A \bullet B = \text{tr}A^T B = \sum_{i,j} a_{ij} b_{ij} \]
- The operator norm of matrix $A$:
  \[ \|A\|^2 := \max_{0 \neq x \in \mathbb{R}^n} \frac{\|Ax\|^2}{\|x\|^2} \]
- The Frobenius norm of matrix $A$:
  \[ \|A\|_f^2 := A \bullet A = \sum_{i,j} a_{ij}^2 \]
Sometimes we use $X = \text{diag}(x)$

- **Eigenvalues and eigenvectors**

  $$Av = \lambda \cdot v$$

- **Perron-Frobenius Theorem**: a real square matrix with positive entries has a unique largest real eigenvalue and the corresponding eigenvector has strictly positive components.

- **Stochastic Matrices**: $A \geq 0$ with $e^T A = e^T$ (Column-Stochastic), or $Ae = e$ (Row-Stochastic), or Doubly-Stochastic if both. It has a unique largest real eigenvalue 1 and corresponding non-negative right or left eigenvector.
Symmetric Matrices

- $S^n$
- The Frobenius norm:
  \[ \|X\|_f = \sqrt{\text{tr}X^TX} = \sqrt{X \cdot X} \]
- Positive Definite (PD): $Q \succ 0$ iff $x^TQx > 0$, for all $x \neq 0$. The sum of PD matrices is PD.
- Positive Semidefinite (PSD): $Q \succeq 0$ iff $x^TQx \geq 0$, for all $x$. The sum of PSD matrices is PSD.
- PSD matrices: $S^n_+$, $\text{int } S^n_+$ is the set of all positive definite matrices.
Known Inequalities

- Cauchy-Schwarz: given \( x, y \in \mathbb{R}^n \), \( x^T y \leq \|x\|\|y\| \).
- Triangle: given \( x, y \in \mathbb{R}^n \), \( \|x + y\| \leq \|x\| + \|y\| \).
- Arithmetic-geometric mean: given \( x \in \mathbb{R}^n_+ \),
  \[
  \frac{\sum x_j}{n} \geq \left( \prod x_j \right)^{1/n}.
  \]
When \( x \) and \( y \) are two distinct points in \( \mathbb{R}^n \) and \( \alpha \) runs over \( \mathbb{R} \),

\[
\{ z : z = \alpha x + (1 - \alpha)y \}
\]

is the line connecting \( x \) and \( y \). When \( 0 \leq \alpha \leq 1 \), it is called the convex combination of \( x \) and \( y \) and it is the line segment between \( x \) and \( y \).

\[
\{ z : z = \alpha x + \beta y \},
\]

for multipliers \( \alpha, \beta \), is the linear combination of \( x \) and \( y \), and it is the hyperplane containing origin and \( x \) and \( y \). When \( \alpha \geq 0, \beta \geq 0 \), it is called the conic combination...
Convex Set

- $\Omega$ is said to be a **convex** set if for every $x^1, x^2 \in \Omega$ and every real number $\alpha \in [0, 1]$, the point $\alpha x^1 + (1 - \alpha) x^2 \in \Omega$.

- **Ball and Ellipsoid**: for given $y \in R^n$ and positive definite matrix $Q$:
  \[ E(y, Q) = \{ x : (x - y)^T Q (x - y) \leq 1 \} \].

- The **intersection** of convex sets is convex, the **sum-set** of convex sets is convex, the **scaled-set** of a convex set is convex.

- The **convex hull** of a set $\Omega$ is the intersection of all convex sets containing $\Omega$. Given column-points of $A$, the convex hull is $\{ z = Ax : e^T x = 1, x \geq 0 \}$.

  **SVM Claim**: two point sets are separable by a plane if any only if their convex hulls are separable.

- An **extreme** point in a convex set is a point that cannot be expressed as a convex combination of other two distinct points of the set.

- A set is **polyhedral** if it has finitely many extreme points; $\{ x : Ax = b, x \geq 0 \}$ and $\{ x : Ax \leq b \}$ are convex polyhedral.
Proof of convex set

• All solutions to the system of linear equations, \( \{x : Ax = b\} \), form a convex set.

• All solutions to the system of linear inequalities, \( \{x : Ax \leq b\} \), form a convex set.

• All solutions to the system of linear equations and inequalities, \( \{x : Ax = b, x \geq 0\} \), form a convex set.

• **Ball** is a convex set: given center \( y \in \mathbb{R}^n \) and radius \( r > 0 \), \( B(y, r) = \{x : \|x - y\| \leq r\} \).

• **Ellipsoid** is a convex set: given center \( y \in \mathbb{R}^n \) and positive definite matrix \( Q \), 
  \( E(y, Q) = \{x : (x - y)^TQ(x - y) \leq 1\} \).
Consider the set $B$ of all $b$, for a fixed $A$, such that the set, $\{x : Ax = b, \ x \geq 0\}$, is feasible.

Show that $B$ is a convex set.

Example:

$$B = \{b : \{(x_1, x_2) : x_1 + x_2 = b, \ (x_1, x_2) \geq 0\} \text{ is feasible}\}.$$
A set $C$ is a cone if $x \in C$ implies $\alpha x \in C$ for all $\alpha > 0$.

The intersection of cones is a cone.

A convex cone is a cone and also a convex set.

A pointed cone is a cone that does not contain a line.

Dual:

\[ C^* := \{ y : x \cdot y \geq 0 \text{ for all } x \in C \}. \]

**Theorem 1** The dual is always a closed convex cone, and the dual of the dual is the closure of convex hall of $C$.
Cone Examples

- **Example 2.1**: The $n$-dimensional non-negative orthant, $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$, is a convex cone. The dual cone is itself.

- **Example 2.2**: The set of all positive semi-definite matrices in $\mathbb{S}^n$, $\mathbb{S}^n_+$, is a convex cone, called the positive semi-definite matrix cone. The dual cone is itself.

- **Example 2.3**: The set $\{x \in \mathbb{R}^n : x_1 \geq \|x_{-1}\|\}$, $\mathcal{N}^n_2$, is a convex cone in $\mathbb{R}^n$ called the second-order cone. The dual cone is itself.

- **Example 2.4**: The set $\{x \in \mathbb{R}^n : x_1 \geq \|x_{-1}\|_p\}$, $\mathcal{N}^n_p$, is a convex cone in $\mathbb{R}^n$ called the $p$-order cone with $p \geq 1$. The dual cone is the $q$-order cone with $\frac{1}{q} + \frac{1}{p} = 1$. 


Polyhedral Convex Cones

- A cone $C$ is (convex) **polyhedral** if $C$ can be represented by
  
  $\{ x : Ax \leq 0 \}$ or $\{ x : x = Ay, \ y \geq 0 \}$

  for some matrix $A$. In the latter case, $K$ is generated by the columns of $A$.

- The nonnegative orthant is a polyhedral cone but the second-order cone is not polyhedral.
Figure 1: Polyhedral and non-polyhedral cones.
Real Functions

- **Continuous functions**

- **Weierstrass theorem**: a continuous function $f$ defined on a compact set (bounded and closed) $\Omega \subset \mathbb{R}^n$ has a minimizer in $\Omega$.

- The **gradient vector**: $\nabla f(x) = \{ \partial f / \partial x_i \}$, for $i = 1, \ldots, n$.

- The **Hessian matrix**: $\nabla^2 f(x) = \left\{ \frac{\partial^2 f}{\partial x_i \partial x_j} \right\}$ for $i = 1, \ldots, n; \ j = 1, \ldots, n$.

- **Vector function**: $\mathbf{f} = (f_1; f_2; \ldots; f_m)$

- The **Jacobian matrix** of $\mathbf{f}$ is

\[
\nabla \mathbf{f}(x) = \begin{pmatrix}
\nabla f_1(x) \\
\vdots \\
\nabla f_m(x)
\end{pmatrix}.
\]
• The least upper bound or supremum of $f$ over $\Omega$

$$\sup\{f(x) : x \in \Omega\}$$

and the greatest lower bound or infimum of $f$ over $\Omega$

$$\inf\{f(x) : x \in \Omega\}$$
Convex Functions

- \( f \) is a (strongly) convex function iff for \( 0 < \alpha < 1 \),

\[
f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y).
\]

- The sum of convex functions is a convex function; the max of convex functions is a convex function;

- The Composed function \( f(\phi(x)) \) is convex if \( \phi(x) \) is a convex and \( f(\cdot) \) is convex&non-decreasing.

- The (lower) level set of \( f \) is convex:

\[
L(z) = \{x : f(x) \leq z\}.
\]

- Convex set \( \{(z; x) : f(x) \leq z\} \) is called the epigraph of \( f \).

- \( tf(x/t) \) is a convex function of \( (t; x) \) for \( t > 0 \) if \( f(\cdot) \) is a convex function; it’s homogeneous with degree 1.
Convex Function Examples

- $\|x\|_p$ for $p \geq 1$.

  \[ \|\alpha x + (1 - \alpha)y\|_p \leq \|\alpha x\|_p + \|(1 - \alpha)y\|_p \leq \alpha\|x\|_p + (1 - \alpha)\|y\|_p, \]

  from the triangle inequality.

- Logistic function $\log(1 + e^{a^T x + b})$ is convex.

- $e^{x_1} + e^{x_2} + e^{x_3}$.

- $\log(e^{x_1} + e^{x_2} + e^{x_3})$: we will prove it later.

**Theorem 2** Every local minimizer is a global minimizer in minimizing a convex objective function over a convex feasible set. If the objective is strongly convex in the feasible set, the minimizer is unique.

**Theorem 3** Every local minimizer is a boundary solution in minimizing a concave objective function (with non-zero gradient everywhere) over a convex feasible set. If the objective is strongly concave in the feasible set, every local minimizer must be an extreme solution.
Proof of convex function

Consider the minimal-objective value function of $b$ for fixed $A$ and $c$:

$$z(b) := \text{minimize} \quad c^T x$$
subject to \quad $A x = b,$
\quad $x \geq 0.$

Show that $z(b)$ is a convex function in $b$ for all feasible $b$. 

Taylor’s theorem or the mean-value theorem:

Theorem 4 Let $f \in C^1$ be in a region containing the line segment $[x, y]$. Then there is an $\alpha$, $0 \leq \alpha \leq 1$, such that

$$f(y) = f(x) + \nabla f(\alpha x + (1 - \alpha)y)(y - x).$$

Furthermore, if $f \in C^2$ then there is an $\alpha$, $0 \leq \alpha \leq 1$, such that

$$f(y) = f(x) + \nabla f(x)(y - x) + (1/2)(y - x)^T \nabla^2 f(\alpha x + (1 - \alpha)y)(y - x).$$

Theorem 5 Let $f \in C^1$. Then $f$ is convex over a convex set $\Omega$ if and only if

$$f(y) \geq f(x) + \nabla f(x)(y - x)$$

for all $x, y \in \Omega$.

Theorem 6 Let $f \in C^2$. Then $f$ is convex over a convex set $\Omega$ if and only if the Hessian matrix of $f$ is positive semi-definite throughout $\Omega$. 
System of Linear Equations

Solve for \( x \in \mathbb{R}^n \) from:

\[
\begin{align*}
a_1x &= b_1 \\
a_2x &= b_2 \\
\vdots &= \vdots \\
a_mx &= b_m
\end{align*}
\Rightarrow Ax = b
\]
Figure 2: System of linear equations

\[3x + 2y = 12\]
\[2x + 3y = 12\]

(2.4, 2.4)
(4, 0) (6, 0)
(0, 4)
(0, 6)
Theorem 7  Given $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, the system $\{x : Ax = b\}$ has a solution if and only if $A^T y = 0$ and $b^T y \neq 0$ has no solution.

A vector $y$, with $A^T y = 0$ and $b^T y \neq 0$, is called an infeasibility certificate for the system.

Example  Let $A = (1; -1)$ and $b = (1; 1)$. Then, $y = (1/2; 1/2)$ is an infeasibility certificate.

Alternative systems: $\{x : Ax = b\}$ and $\{y : A^T y = 0, b^T y \neq 0\}$. 
Figure 3: \( \mathbf{b} \) is not in the set \( \{ A\mathbf{x} : \mathbf{x} \} \), and \( \mathbf{y} \) is the distance vector from \( \mathbf{b} \) to the set.
Linear least-squares problem

Given \( A \in \mathbb{R}^{m \times n} \) and \( c \in \mathbb{R}^n \),

\[
(\text{LS}) \quad \text{minimize} \quad \| c - A^T y \|^2 \\
\text{subject to} \quad y \in \mathbb{R}^m.
\]

A close form solution:

\[
AA^T y = Ac \quad \text{or} \quad y = (AA^T)^{-1} Ac.
\]

\[
\begin{align*}
\text{c} - A^T y &= \text{c} - A^T (AA^T)^{-1} Ac \\
&= \text{c} - P\text{c}
\end{align*}
\]

Projection matrix: \( P = A^T (AA^T)^{-1} A \) or \( P = I - A^T (AA^T)^{-1} A \).
Figure 4: Projection of $\mathbf{c}$ onto a subspace
Choleski decomposition method

\[ AA^T = L \Lambda L^T \]

\[ L \Lambda L^T y^* = Ac \]
System of nonlinear equations

Given $f(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, the problem is to solve $n$ equations for $n$ unknowns:

$$f(x) = 0.$$ 

Given a point $x^k$, Newton’s Method sets

$$f(x) \approx f(x^k) + \nabla f(x^k)(x - x^k) = 0.$$ 

$$x^{k+1} = x^k - (\nabla f(x^k))^{-1}f(x^k)$$

or solve for direction vector $d_x$:

$$\nabla f(x^k)d_x = -f(x^k) \quad \text{and} \quad x^{k+1} = x^k + d_x.$$
Figure 5: Newton’s method for root finding
The quasi Newton method

For minimization of objective function $f(x)$, then $f(x) = \nabla f(x)$

$$x^{k+1} = x^k - \alpha (\nabla^2 f(x^k))^{-1} \nabla f(x^k)$$

where scalar $\alpha \geq 0$ is called step-size. More generally

$$x^{k+1} = x^k - \alpha M^k \nabla f(x^k)$$

where $M^k$ is an $n \times n$ symmetric matrix. In particular, if $M^k = I$, the method is called the gradient method, where $f$ is viewed as the gradient vector of a real function.
• \( \{x^k\}_{0}^{\infty} \) denotes a sequence \( x^0, x^1, x^2, \ldots, x^k, \ldots \).

• \( x^k \to \bar{x} \) iff

\[
\|x^k - \bar{x}\| \to 0
\]

• \( g(x) \geq 0 \) is a real valued function of a real nonnegative variable, the notation \( g(x) = O(x) \) means that \( g(x) \leq \bar{c}x \) for some constant \( \bar{c} \);

• \( g(x) = \Omega(x) \) means that \( g(x) \geq c x \) for some constant \( c \);

• \( g(x) = \theta(x) \) means that \( cx \leq g(x) \leq \bar{c}x \).

• \( g(x) = o(x) \) means that \( g(x) \) goes to zero faster than \( x \) does:

\[
\lim_{x \to 0} \frac{g(x)}{x} = 0
\]