Dual Interpretations and Duality Applications

Yinyu Ye
Department of Management Science and Engineering
Stanford University
Stanford, CA 94305, U.S.A.

http://www.stanford.edu/~yyye

(LY, Chapters 4.1-4.2, 6.8)

Production Problem I

$$\max\ \mathbf{p}^T\mathbf{x} \quad \text{s.t.} \quad A\mathbf{x} \le \mathbf{r}, \quad \mathbf{x} \ge \mathbf{0}$$

where

- $\mathbf{p}$: profit margin vector
- A: resources consumption rate matrix
- $\mathbf{r}$: available resource vector
- $\mathbf{x}$: production level decision vector

Production Problem II: Liquidation Pricing

- $\mathbf{y}$: the fair price vector
- $A^T\mathbf{y} \geq \mathbf{p}$: competitiveness
- $\mathbf{y} \geq 0$: positivity
- $\min\ \mathbf{r}^T\mathbf{y}$: minimize the total liquidation cost

$$
\text{Primal}: \quad \begin{array}{llll}
\text{maximize} & x_1 & +2x_2 & \\
\text{subject to} & x_1 & & \leq 1 \\
& & x_2 & \leq 1 \\
& x_1 & +x_2 & \leq 1.5 \\
& x_1, & x_2 & \geq 0.
\end{array}
$$

$$
\text{Dual}: \quad \begin{array}{lllll}
\text{minimize} & y_1 & +y_2 & +1.5y_3 & \\
\text{subject to} & y_1 & & +y_3 & \geq 1 \\
& & y_2 & +y_3 & \geq 2 \\
& y_1, & y_2, & y_3 & \geq 0.
\end{array}
$$

Optimal Value Function

For fixed matrix A and objective coefficient vector $\mathbf{c}$, the optimal value is a function of right-hand-side vector $\mathbf{b}$:

$$
\begin{array}{rll}
z(\mathbf{b}) = & \text{minimize} & \mathbf{c}^T\mathbf{x} \\
& \text{subject to} & A\mathbf{x} \;= \mathbf{b}, \\
& & \mathbf{x} \;\geq\; \mathbf{0}.
\end{array}
$$

Theorem: $z(\mathbf{b})$ is a convex function in $\mathbf{b}$, that is, for any $0 \leq \alpha \leq 1$

$$z(\alpha\mathbf{b}^1 + (1-\alpha)\mathbf{b}^2) \leq \alpha z(\mathbf{b}^1) + (1-\alpha)z(\mathbf{b}^2).$$

Shadow Prices of the Optimal Value

Suppose a new right-hand-vector $\mathbf{b}^+$ such that

$$b_k^+ = b_k + \delta \quad \text{and} \quad b_i^+ = b_i, \ \forall i \neq k.$$

Then, the optimal dual solution $\mathbf{y}^*$ has a property

$$y_k^* = \frac{z(\mathbf{b}^+) - z(\mathbf{b})}{\delta}$$

as long as $\mathbf{y}^*$ remains the dual optimal solution for $\mathbf{b}^+$, since

$$z(\mathbf{b}^+) = (\mathbf{b}^+)^T \mathbf{y}^* = z(\mathbf{b}) + \delta \cdot y_k^*.$$

Thus, the optimal dual solution is the shadow price vector of the right-hand-vector vector, or the rate of the net change of the optimal objective value over the net change of an entry of the right-hand-vector vector.

Transportation Problem

$$
\begin{array}{lll}
\min & \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} & \\
\text{s.t.} & \sum_{j=1}^{n} x_{ij} & = s_i, \ \forall i = 1, ..., m \\
& \sum_{i=1}^{m} x_{ij} & = d_j, \ \forall j = 1, ..., n \\
& x_{ij} & \geq 0, \ \forall i, j.
\end{array}
$$

Transportation Dual

$$\begin{array}{lll} \max & \sum_{i=1}^{m} s_i u_i + \sum_{j=1}^{n} d_j v_j & \\ \text{s.t.} & u_i + v_j & \leq c_{ij}, \ \forall i, j. \end{array}$$

u_i: supply site unit price

v_i: demand site unit price

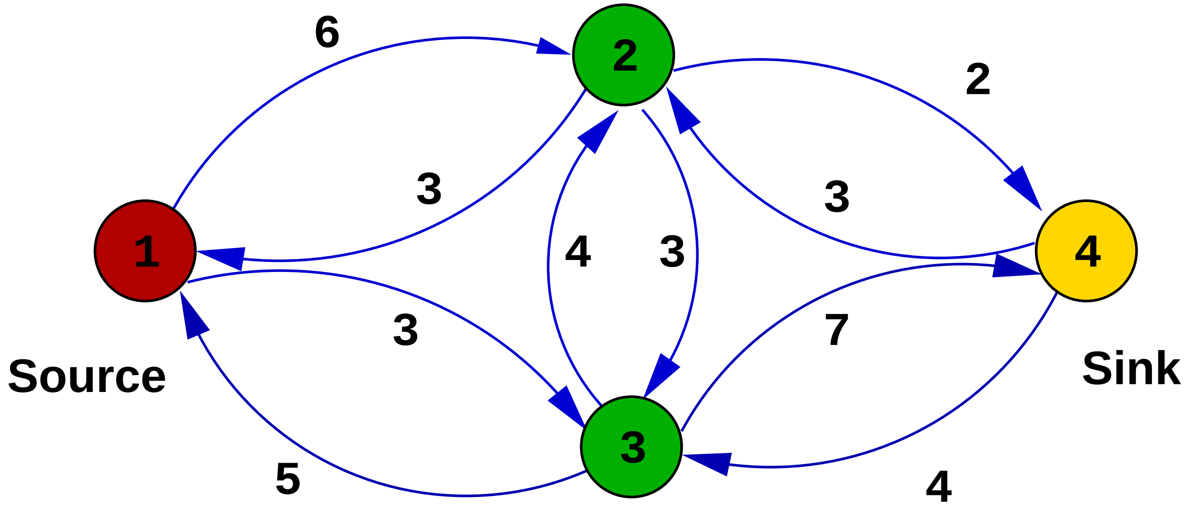
$u_i + v_j \leq c_{ij}$: competitiveness

Max-Flow and Min-Cut

Given a directed graph with nodes $1, ..., m$ and edges $\mathcal{A}$, where node 1 is called source and node m is called the sink, and each edge (i, j) has a flow rate capacity k_{ij}. The Max-Flow problem is to find the largest possible flow rate from source to sink.

Let x_{ij} be the flow rate from node i to node j. Then the problem can be formulated as

$$
\begin{array}{ll}
\text{maximize} & x_{m1} \\
\text{subject to} & \sum_{j:(j,1)\in\mathcal{A}} x_{j1} - \sum_{j:(1,j)\in\mathcal{A}} x_{1j} + x_{m1} = 0, \\
& \sum_{j:(j,i)\in\mathcal{A}} x_{ji} - \sum_{j:(i,j)\in\mathcal{A}} x_{ij} = 0, \forall i = 2, ..., m-1, \\
& \sum_{j:(j,m)\in\mathcal{A}} x_{jm} - \sum_{j:(m,j)\in\mathcal{A}} x_{mj} - x_{m1} = 0, \\
& 0 \le x_{ij} \le k_{ij}, \ \forall (i,j) \in \mathcal{A}.
\end{array}
$$


6
2
2
3
3
1
4
4
3
3
7
Source
Sink
3
5
4

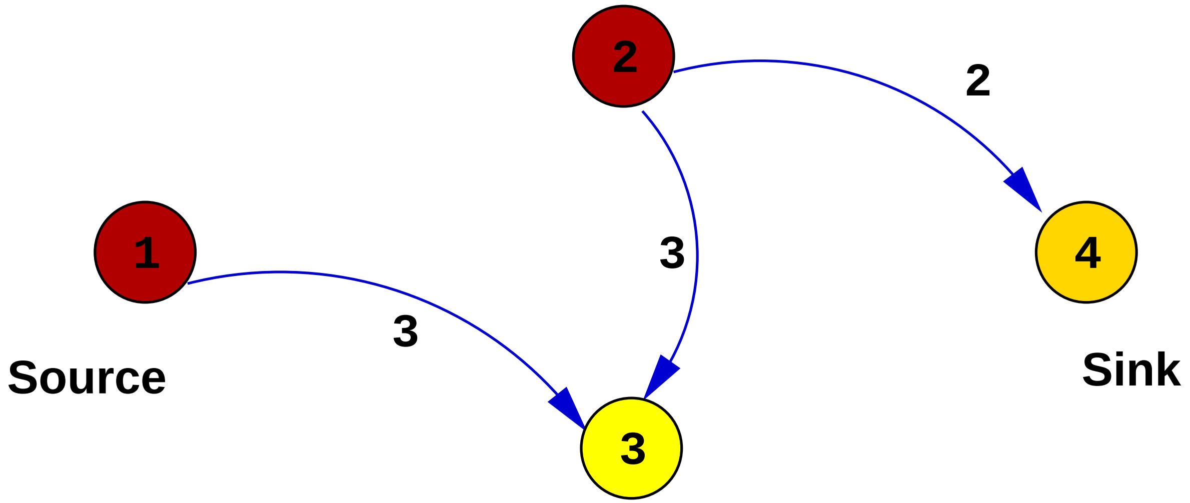
The dual of the Max-Flow problem

$$\begin{array}{ll}\text{minimize} & \sum_{(i,j)\in\mathcal{A}} k_{ij} z_{ij} \\ \text{subject to} & -y_i + y_j + z_{ij} \geq 0, \ \forall (i,j) \in \mathcal{A}, \\ & y_1 - y_m = 1, \\ & z_{ij} \geq 0, \ \forall (i,j) \in A. \end{array}$$

y_i: node potential value. At an optimal solution has property $y_1 = 1, \ y_m = 0$ and for all other i:

$$y_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$$

This problem is called the Min-Cut problem.


2
2
1
3
3
4
Source
3
Sink

Application: Combinatorial Auction Pricing I

Given the m different states that are mutually exclusive and exactly one of them will be true at the maturity. A contract on a state is a paper agreement so that on maturity it is worth a notional $\$1$ if it is on the winning state and worth $\$0$ if is not on the winning state. There are n orders betting on one or a combination of states, with a price limit and a quantity limit.

Combinatorial Auction Pricing II: an order

The jth order is given as $(\mathbf{a}_j \in R^m_+,\ \pi_j \in R_+,\ q_j \in R_+)$: $\mathbf{a}_j$ is the combination betting vector where each component is either 1 or 0

$$\mathbf{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \dots \\ a_{mj} \end{pmatrix},$$

where 1 is winning and 0 is non-winning; π_j is the price limit for one such a contract, and q_j is the maximum number of contracts the better like to buy.

World Cup Information Market

Order:	$\#1$	$\#2$	$\#3$	$\#4$	$\#5$
Argentina	1	0	1	1	0
Brazil	1	0	0	1	1
Italy	1	0	1	1	0
Germany	0	1	0	1	1
France	0	0	1	0	0
Bidding Prize: π	0.75	0.35	0.4	0.95	0.75
Quantity limit: $\mathbf{q}$	10	5	10	10	5
Order fill: $\mathbf{x}$	x_1	x_2	x_3	x_4	x_5

Combinatorial Auction Pricing III: Pricing each state

Let x_j be the number of contracts awarded to the jth order. Then, the jth better will pay the amount

$$\pi_j \cdot x_j$$

and the total collected amount is

$$\sum_{j=1}^{n} \pi_j \cdot x_j = \pi^T \mathbf{x}$$

If the ith state is the winning state, then the auction organizer need to pay back

$$\left(\sum_{j=1}^{n} a_{ij} x_j \right)$$

The question is, how to decide $\mathbf{x} \in R^n$.

Combinatorial Auction Pricing IV: LP model

$$\begin{array}{lrl} \max & \pi^T \mathbf{x} - z & \\ \text{s.t.} & A\mathbf{x} - \mathbf{e} \cdot z & \leq \mathbf{0}, \\ & \mathbf{x} & \leq \mathbf{q}, \\ & \mathbf{x} & \geq 0. \end{array}$$

$\pi^T \mathbf{x}$: the optimistic amount can be collected. z: the worst-case amount need to pay back.

Combinatorial Auction V: The dual

$$\begin{array}{lrl} \min & \mathbf{q}^T \mathbf{y} & \\ \text{s.t.} & A^T \mathbf{p} + \mathbf{y} & \geq \pi, \\ & \mathbf{e}^T \mathbf{p} & = 1, \\ & (\mathbf{p}, \mathbf{y}) & \geq 0. \end{array}$$

$\mathbf{p}$ represents the state price and What is $\mathbf{y}$?

Combinatorial Auction V: Strict Complementarity

$x_j > 0$	$\mathbf{a}_j^T\mathbf{p} + y_j = \pi_j$ and $y_j \geq 0$ so that $\mathbf{a}_j^T\mathbf{p} \leq \pi_j$
$0 < x_j < q_j$	$y_j = 0$ so that $\mathbf{a}_j^T\mathbf{p} = \pi_j$
$x_j = q_j$	$y_j > 0$ so that $\mathbf{a}_j^T\mathbf{p} < \pi_j$
$x_j = 0$	$\mathbf{a}_j^T\mathbf{p} + y_j > \pi_j$ and $y_j = 0$ so that $\mathbf{a}_j^T\mathbf{p} > \pi_j$

The price is Fair:

$$\mathbf{p}^T(A\mathbf{x} - \mathbf{e}\cdot z) = 0 \quad \text{implies} \quad \mathbf{p}^T A\mathbf{x} = \mathbf{p}^T\mathbf{e}\cdot z = z;$$

that is, the worst case cost equals the worth of total shares. Moreover, if a lower bid wins the auction, so does the higher bid on any same type of bids.

World Cup Information Market Result

Order:	$\#1$	$\#2$	$\#3$	$\#4$	$\#5$	State Price
Argentina	1	0	1	1	0	0.2
Brazil	1	0	0	1	1	0.35
Italy	1	0	1	1	0	0.2
Germany	0	1	0	1	1	0.25
France	0	0	1	0	0	0
Bidding Price: π	0.75	0.35	0.4	0.95	0.75	
Quantity limit: $\mathbf{q}$	10	5	10	10	5	
Order fill: $\mathbf{x}^*$	5	5	5	0	5	

Question 1: The uniqueness of dual prices?

Combinatorial Auction Pricing VI: convex programming model

$$
\begin{array}{lrl}
\max & \pi^T \mathbf{x} - z + u(\mathbf{s}) & \\
\text{s.t.} & A\mathbf{x} - \mathbf{e} \cdot z + \mathbf{s} & = \mathbf{0}, \\
& \mathbf{x} & \leq \mathbf{q}, \\
& \mathbf{x},\ \mathbf{s} & \geq 0.
\end{array}
$$

$u(\mathbf{s})$: a value function for the market organizer on slack shares.

If $u(\cdot)$ is a strictly concave function, then the state price vector is unique.

Question 2: Online allocation?

Online Resource Allocation Linear Programming

$$
\begin{array}{ll}
\text{maximize}_{\mathbf{x}} & \sum_{j=1}^{n} \pi_j x_j \\
\text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \; \forall\, i = 1, 2, ..., m, \\
& 0 \leq x_j \leq 1, \; \forall\, j = 1, ..., n.
\end{array}
$$

Approach 1 (SCPM):

$$
\begin{array}{ll}
\text{maximize}_{x_k, \mathbf{s}} & \pi_k x_k + u(\mathbf{s}) \\
\text{s.t.} & a_{ik} x_k + s_i = b_i - \sum_{j=1}^{k-1} a_{ik} \bar{x}_k, \; \forall\, i = 1, 2, ..., m, \\
& 0 \leq x_k \leq 1, \\
& s_i \geq 0, \; \forall\, i = 1, ..., m.
\end{array}
$$

Approach 2 (SLPM):

$$
\begin{array}{ll}
\text{maximize}_{x_1, ..., x_k} & \sum_{j=1}^{k} \pi_i x_i \\
\text{s.t.} & \sum_{j=1}^{k} a_{ik} x_k \leq \frac{k}{n} b_i, \; \forall\, i = 1, 2, ..., m, \\
& 0 \leq x_j \leq 1, \; \forall\, j = 1, ..., k.
\end{array}
$$

and use the dual prices of the partial LP for decisions for the next period ...

Online Resource Allocation with Production Costs

One may consider more general resource allocation problems with production costs:

$$
\begin{array}{ll}
\text{maximize}_{\mathbf{x}} & \sum_{j=1}^{n}(\pi_j x_j - \sum_k c_{ijk} y_{ijk}) \\
\text{s.t.} & \sum_k y_{ijk} = a_{ij} x_j;\ \forall i, j, \\
 & \sum_{i,j} y_{ijk} \le c_k;\ \forall k, \\
 & 0 \le x_j \le 1,\ \forall\, j = 1, ..., n;
\end{array}
$$

where c_{ijk} is the cost allocate good/resource i, which is produced by producer $k = 1, ..., K$, to bidder j; and c_k is the production capacity of producer k.