**Z-Matrices**

**Definition.** A matrix $M \in \mathbb{R}^{n \times n}$ is said to be a $Z$-matrix if all of its off-diagonal entries (if any) are nonpositive: $m_{ij} \leq 0$, for all $i \neq j$.

The class $Z$ of such matrices is complete.

The diagonal entries of a $Z$-matrix are not sign-restricted, but the most interesting and important results are obtained for matrices $M$ belonging to

$$K := P \cap Z.$$
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**Minkowski matrices**

Related to $K$-matrices are the so-called $M$-matrices. These are of the form

$$A = sI - B, \quad s \geq \rho(B), \quad B \geq 0 \text{ (elementwise)}$$

where $\rho(B)$ is the spectral radius of $B$. 


Some important seminal references on this subject are:


In their 1962 paper, Fiedler and Pták gave 13 equivalent conditions for a $Z$-matrix to belong to $P$ and hence to $K$.

There are many more. For example, in


the authors list 50 such conditions that are equivalent to the statement “$A$ is a nonsingular $M$-matrix.” And the list is not complete!
Examples.

(a) The matrix $M$ in the problem of finding the convex hull of a finite set of points in the plane is a symmetric (tridiagonal) $K$-matrix.

Definition. Symmetric $K$-matrices are called Stieltjes-matrices.

The form of the matrix above is the same as that which arises in the isotone regression problem

$$\begin{align*}
\text{minimize} \quad & \sum_{i=0}^{n} d_i (x_i - a_i)^2 \\
\text{subject to} \quad & x_0 \leq x_1 \leq \cdots \leq x_n
\end{align*}$$

where $d_i > 0$ and $a_i$ is arbitrary for all $i$.

This problem is studied in the paper

(b) In Section 7.3 of his book *Introduction to Stochastic Processes*, E. Çinlar, discusses an optimal stopping (game) problem which leads to a linear program stated as

\[
\begin{align*}
\text{minimize} & \quad e^T v \\
\text{subject to} & \quad v \geq \alpha P v \quad (\alpha \in [0, 1]) \\
& \quad v \geq f \\
& \quad v \geq 0
\end{align*}
\]

The matrix $P$ is square and row stochastic: the elements of each row are nonnegative and sum to 1. If $g := \max(f, 0)$, the last two constraints become $v \geq g$.

Now define $z := v - g$. Then the problem becomes

\[
\begin{align*}
\text{minimize} & \quad e^T z + e^T g \\
\text{subject to} & \quad (I - \alpha P)g + (I - \alpha P)z \geq 0 \\
& \quad z \geq 0
\end{align*}
\]
We ignore the additive constant $e^T g$ in the objective function and define

$$q = (I - \alpha P)g \quad \text{and} \quad M = I - \alpha P.$$ 

Then certainly $M \in \mathbb{Z}$ and possibly (depending on $\alpha$) $M \in \mathbb{K}$.

The constraints

$$\begin{align*}
(I - \alpha P)g + (I - \alpha P)z & \geq 0 \\
z & \geq 0
\end{align*}$$

become just those of the LCP $(q, M)$.

But the LP is not an LCP.

We will see that the two problems are equivalent.
Least element theory of polyhedral sets

Definition. A set $S \subseteq \mathbb{R}^n$ is a *meet semi-sublattice* under the componentwise ordering of $\mathbb{R}^n$ if

$$
\text{for all } x, y \in S, \quad z = \min(x, y) \in S.
$$

The vector $z$ is called the *meet* of $x$ and $y$.

Proposition. If $M \in \mathbb{R}^{n \times n} \cap \mathbb{Z}$ and $q \in \mathbb{R}^n$, then $FEA(q, M)$ is a meet semi-sublattice.

Proof. This is a routine exercise.
Definition. A set $S \subseteq \mathbb{R}^n$ is bounded below if there exists a vector $u \in \mathbb{R}^n$ such that $x \geq u$ for all $x \in S$. If $u \in S$, then $u$ is the least element of $S$.

A set that is bounded below need not have a least element. But if a least element exists, it must be unique.
Theorem. If $S \subseteq \mathbb{R}^n$ is a nonempty meet semi-sublattice that is closed and bounded below, then $S$ has a least element.

Proof. Let $p \in \mathbb{R}^n_{++}$ and form the optimization problem

$$\text{minimize } p^T x \text{ subject to } x \in S.$$ 

(If $S$ is polyhedral, this is a linear program.) Let $x' \in S$ be arbitrary. The problem above is equivalent to

$$\text{minimize } p^T x \text{ subject to } x \in S, x \geq u, p^T x \leq p^T x'.$$

This problem has a nonempty compact feasible region and hence an optimal solution, $\hat{x}$. This must be the least element of $S$, for if $x \in S$ is arbitrary, then $z = \min(x, \hat{x}) \in S$. By definition of $\hat{x}$, $p^T \hat{x} \leq p^T z$. But $z \leq \hat{x}$ and $p > 0$; thus we have $p^T z \leq p^T \hat{x}$. Hence $\hat{x} = z$.
Remark. The following theorem implies that $\mathbb{Z} \subset \mathbb{Q}_0$.

**Theorem.** If $(q, M)$ is a feasible LCP with $M \in \mathbb{Z}$, then $\text{FEA}(q, M)$ contains a least element $u \in \text{SOL}(q, M)$.

**Proof.** Since $M \in \mathbb{Z}$, $S := \text{FEA}(q, M)$ is a meet semi-sublattice. It is nonempty, closed and bounded below, so it has a least element, $u$.

We need to prove $u_i(q + Mu)_i = 0$ for all $i$.

Suppose there is an $i$ for which $u_i(q + Mu)_i > 0$. Define $z = u - \delta I_i$. It is easy to show that for sufficiently small $\delta > 0$, we have $z \leq u$ and $z \in \text{FEA}(q, M)$. This contradicts the least element property of $u$.

Remark. When $M \in \mathbb{Z}$, the linear complementarity problem $(q, M)$ can be treated by a *linear programming* algorithm such as the simplex method.
Question. Is the least element solution of a feasible LCP $(q, M)$ with $M \in \mathbb{Z}$ the only solution of the problem?
**Question.** Is the least element solution of a feasible LCP \((q, M)\) with \(M \in \mathbb{Z}\) the only solution of the problem?

**Answer.** No. Consider the example where

\[
M = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\]

The feasible region is the halfline

\[
\{ z : (z_1, z_2) = (1, 0) + \theta(1, 1), \ \theta \geq 0 \}
\]

every point of which solves \((q, M)\)
**Theorem.** Let \( M \in R^{n \times n} \). Then \( M \in \mathbf{Z} \) if and only if for every \( q \in R^n \), the set \( \text{FEA}(q, M) \) contains a least element that solves \((q, M)\).

**Proof.** We have already proved the sufficiency part. To prove the necessity, suppose that \( m_{ij} > 0 \) for some \( i \neq j \). Define \( q = I_j - M.j \). Then \( I_j \in \text{FEA}(q, M) \). Let \( x \) be the least element of \( \text{FEA}(q, M) \). Then \( 0 \leq x \leq I_j \). So \( x_k = 0 \) if \( k \neq j \). In particular, for \( i \) (which does not equal \( j \))

\[
0 \leq (q + Mx)_i = -m_{ij} + \sum_{k=1}^{n} m_{ik}x_k \\
= -m_{ij} + m_{ij}x_j = m_{ij}(x_j - 1) \leq 0
\]

Hence \( x_j = 1 \) and \( x = I_j \). Since \( x \in \text{SOL}(q, M) \), we have

\[
0 = x_j(q + Mx)_j = (q + Mx)_j = 1.
\]

This contradiction shows that \( M \in \mathbf{Z} \).
**Corollary.** Let $M \in \mathbb{R}^{n \times n}$. Then $M \in \mathbf{K}$ if and only if for all $q \in \mathbb{R}^n$, the set $\text{FEA}(q, M)$ contains a least element which is the unique solution of $(q, M)$.

**Remark.** This corollary gives a characterization of the class $\mathbf{K} = \mathbf{P} \cap \mathbf{Z}$ in terms of the LCP.

Recall that when $M \in \mathbf{P}$, we have $|\text{SOL}(q, M)| = 1$ for all $q$. Thus, given the $\mathbf{P}$-matrix $M$, there is a mapping $q \mapsto z(q)$, the unique solution of $(q, M)$.

This notation will prove useful in the next result.
**Proposition.** Let $M$ be a $\mathbf{P}$-matrix of order $n$. Then $M \in \mathbf{K}$ if and only if

\[ [q^1, q^2 \in R^n, \quad q^1 \geq q^2] \implies [z(q^1) \leq z(q^2)].\]

(The solution mapping is antitone.)

**Proof.** If $M \in \mathbf{K}$ and $q^1 \geq q^2$, then $z(q^2) \in \text{FEA}(q^1, M)$. Hence $z(q^1) \leq z(q^2)$.

Conversely, if $z(q)$ has the antitonicity property and $m_{ij} > 0$ for some $i \neq j$, define

\[ q^1 = I_{.j} - M_{.j} \quad \text{and} \quad q^2 = -M_{.j}. \]

Then $q^1 \geq q^2$. Clearly we have $z(q^2) = I_{.j}$.

Moreover, $z(q^1) \leq I_{.j}$ since $I_{.j} \in \text{FEA}(q^1, M)$.

This leads to a contradiction as in the previous theorem.
Theorem. If $M \in R^{n \times n} \cap Z$, the following are equivalent.

(a) $M \in K$.
(b) All *leading* principal minors of $M$ are positive.
(c) $M^{-1}$ exists and is (elementwise) nonnegative.
(d) $M \in S$.
(e) $M \in \bar{S}$.

Proof. (a) $\implies$ (b). Obvious.

(b) $\implies$ (c). This follows by induction and an elementary argument using principal pivoting and the Schur complement.

(c) $\implies$ (d). Choose any $p \in R_{++}^n$ and define $x = M^{-1}p$. Then $x \geq 0$. (In fact $x > 0$, but this requires more discussion.) Thus, $Mx \geq 0, \ x \geq 0$ has a solution, and hence $M \in S$. 
(d) $\implies$ (e). Let $x > 0$ satisfy $Mx > 0$. Then clearly, for any $\alpha \subseteq \{1, \ldots, n\}$ we have $x_\alpha > 0$ and $M_{\alpha\alpha}x_\alpha > 0$, so $M \in \bar{S}$.

(e) $\implies$ (a). Assume $M \in \bar{S}$. Then for any $q$ the LCP $(q, M)$ will be feasible. But $M$ is also in $\mathbb{Z}$, so a least element solution $u$ must exist. To complete the proof, it suffices to show that $u$ is the only solution.

Let $\tilde{z}$ be any solution of $(q, M)$. Then $\tilde{z} \geq u$. Define the vector $v = q + Mu \geq 0$. Then $\tilde{z} - u \geq 0$ solves $(v, M)$ because

$$v + M(\tilde{z} - u) = (q + Mu) + M\tilde{z} - Mu = q + M\tilde{z} \geq 0$$

and

$$0 \leq (\tilde{z} - u)^T[q + Mu + M(\tilde{z} - u)] = -u^T[q + M\tilde{z}] \leq 0.$$

However, this LCP has the unique solution 0, because $v \geq 0$ and $M \in \bar{S} = E$. Hence $\tilde{z} = u$. 