Equilibrium Problems

The first 14 of these slides are based on the paper


As originally written, this work is set in a real topological space $X$. The space $X^*$ denotes the topological dual of $X$ and is topologized so that the bilinear form $\langle \cdot, \cdot \rangle$ is continuous on $X^* \times X$.

We choose to work in the finite-dimensional space $\mathbb{R}^n$ which is its own dual.
Let $K$ be a given nonempty subset of $\mathbb{R}^n$, and let $f : K \times K \to \mathbb{R}$ be a given function satisfying the property

$$f(x, x) = 0 \quad \forall x \in K.$$  

**Definition.** Given $f$ and $K$ as above, *equilibrium problem* (EP) denoted $(f, K)$ is understood to be that of finding $\bar{x} \in \mathbb{R}^n$ such that

$$f(\bar{x}, y) \geq 0 \quad \text{for all } y \in K. \quad \text{(EP)}$$

The set $K$ is often assumed to be closed and convex.

The function $f$ (above) is called *monotone* if

$$f(x, y) + f(y, x) \leq 0 \quad \text{for all } y \in K.$$
Examples

1. Optimization as an equilibrium problem

Let \( \varphi : K \rightarrow R \) be given. We seek \( \bar{x} \in K \) such that

\[
\varphi(\bar{x}) \leq \varphi(y) \quad \text{for all } y \in K.
\]  

(1)

It is immediately seen that \( \bar{x} \) solves the optimization problem (1) if and only if it solves the equilibrium problem relative to \((f, K)\) where

\[
f(x, y) = \varphi(y) - \varphi(x).
\]

Note that for this function \( f \) we have

\[
f(x, y) + f(y, x) = 0,
\]

hence it is monotone.
2. Saddle point problem as an equilibrium problem

Let \( \varphi : K_1 \times K_2 \to \mathbb{R} \). Then \((\bar{x}_1, \bar{x}_2) \in K_1 \times K_2\) is a saddle point if and only if

\[
\varphi(\bar{x}_1, y_2) \leq \varphi(y_1, \bar{x}_2) \quad \forall (y_1, y_2) \in K_1 \times K_2.
\]

(2)

In particular, if \((\bar{x}_1, \bar{x}_2)\) is a saddle point, we have

\[
\varphi(\bar{x}_1, y_2) \leq \varphi(\bar{x}_1, \bar{x}_2) \leq \varphi(y_1, \bar{x}_2) \quad \forall (y_1, y_2) \in K_1 \times K_2.
\]

To formulate this as an equivalent equilibrium problem, define \( K = K_1 \times K_2 \) and write

\[
f((x_1, x_2), (y_1, y_2)) = \varphi(y_1, x_2) - \varphi(x_1, y_2).
\]

This function is also monotone since it satisfies

\[
f((x_1, x_2), (y_1, y_2)) + f((y_1, y_2), (x_1, x_2)) \equiv 0.
\]
3. Nash equilibrium point problem as an equilibrium point problem
Suppose we have a finite set $\mathcal{I}$ of, say $N$, players.
Let $K_i$ denote the (finite) set of pure strategies of player $i \in \mathcal{I}$.
Define
\[ K = \prod_{i \in \mathcal{I}} K_i. \]
For all $i \in \mathcal{I}$, let $f_i : K \to R$ denote the loss function of player $i$.
For arbitrary $x = (x_1, \ldots, x_N) \in K$ we define
\[ x^i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N) \]
with the obvious modifications for the cases $i = 1$ and $i = N$. 
Definition. An $N$-tuple $\bar{x} \in K$ is a *Nash equilibrium point* if for all $i \in I$

$$f_i(\bar{x}) \leq f_i(\bar{x}^i, y_i) \quad \text{for all } y_i \in K_i. \quad (3)$$

Now define $f : K \times K \to R$ by

$$f(x, y) = \sum_{i \in I} \left( f_i(x^i, y_i) - f_i(x) \right).$$

It can be shown that $\bar{x}$ is a Nash equilibrium point (solution of (3)) if and only if it is an a solution of the equilibrium, problem $(f, K)$. 
4. Fixed point problems as equilibrium problems

Definition. Let \( T : K \to K \) is a given mapping, then \( \bar{x} \in K \) is a fixed point of \( T \) if and only if
\[
\bar{x} = T\bar{x}.
\] (4)

To formulate this as an equilibrium problem, define
\[
f(x, y) = \langle x - Tx, y - x \rangle.
\]

It is easy to see that \( \bar{x} \) is a fixed point of \( T \) (solution of (4)) if and only if it is a solution of \((f, K)\).
Remark. The function

\[ f(x, y) = \langle x - Tx, y - x \rangle \]

is monotone if and only if

\[ \langle Tx - Ty, x - y \rangle \leq \|x - y\|^2 \quad \forall x, y \in K. \]

This amounts to saying that \( f \) is monotone if and only if it is nonexpansive.
5. Convex differentiable optimization as an equilibrium problem

The familiar problem

\[
\begin{align*}
\text{minimize} & \quad \varphi(x) \\
\text{subject to} & \quad x \in K
\end{align*}
\]  

(5)

in which \(\varphi : \mathbb{R}^n \rightarrow \mathbb{R}\) is convex and differentiable has a special optimality criterion.

A vector \(\bar{x}\) solves (5) if and only it it solves the variational inequality problem: Find \(\bar{x} \in K\) such that

\[
\langle \nabla \varphi(\bar{x}), y - \bar{x} \rangle \geq 0 \quad \text{for all } y \in K.
\]

By putting \(f(x, y) = \langle \nabla \varphi(x), y - x \rangle\) for all \(x, y \in K\) we see that this variational inequality problem and the \((\EP) (f, K)\) have the same set of solutions.
Remark. It is well known that because $\varphi$ is convex and differentiable, its gradient $\nabla \varphi$ is a *monotone mapping* in the sense that

$$\langle \nabla \varphi(x) - \nabla \varphi(y), x - y \rangle \geq 0 \quad \text{for all } x, y \in K.$$ 

Now note that with $f(x, y) := \langle \varphi(x), y - x \rangle$, we have

$$f(x, y) + f(y, x) = \langle \nabla \varphi(x), y - x \rangle + \langle \nabla \varphi(y), x - y \rangle$$

$$= - \langle \nabla \varphi(x) - \nabla \varphi(y), x - y \rangle \leq 0.$$ 

Hence in this case, the function $f$ is monotone.
6. Variational operator inequalities as equilibrium problems

Let $T : K \to \mathbb{R}^n$ be a given mapping.

The problem now is to find $\bar{x} \in K$ such that

$$\langle T\bar{x}, y - \bar{x} \rangle \geq 0 \quad \text{for all } y \in K. \quad (6)$$

Here we take $f(x, y) = \langle Tx, y - x \rangle$ and the equivalence of (6) and the corresponding (EP) is trivial.
7. Complementarity problems as equilibrium problems

Let $K$ be a closed convex cone and let $K^*$ denote the polar cone

$$K^* = \{ x^* : \langle x^*, y \rangle \geq 0 \text{ for all } y \in K \}$$

Let $T : K \rightarrow R^n = (R^n)^*$. The complementarity problem $(T, K)$ is defined as that of finding

$$\bar{x} \in K \text{ such that } T\bar{x} \in K^* \text{ and } \langle T\bar{x}, \bar{x} \rangle = 0.$$  \hspace{1cm} (7)

It is well known that when $K$ is a cone, the complementarity problem and the variational inequality problem have the same solutions (if any). This being so, the complementarity problem can be regarded as an equilibrium problem.
8. Variational inequalities with set-valued mappings as equilibrium problems

Suppose we are given $T : K \rightarrow R^n$ with the property that for all $x \in K$ the image $Tx$ is nonempty, convex, and compact subset of $R^n$.

The variational inequality problem with the set-valued mapping $T$ is to solve

$$\bar{x} \in K, \quad \bar{\xi} \in T\bar{x}, \quad \langle \bar{\xi}, y - \bar{x} \rangle \geq 0 \quad \text{for all } y \in K. \quad (8)$$

Now define

$$f(x, y) = \max_{\xi \in Tx} \langle \xi, y - x \rangle.$$

It is easy to see that if $\bar{x}, \bar{\xi}$ is a solution to (8), then

$$f(\bar{x}, y) \geq \langle \bar{\xi}, y - \bar{x} \rangle \geq 0 \quad \text{for all } y \in K$$

so that $\bar{x}$ solves (EP).
The converse direction, namely that if $\bar{x}$ is a solution of (EP) then there exists a suitable element of $T\bar{x}$ such that the two of them together give a solution to (8) is more difficult.
Introduction to Variational Inequalities and Nonlinear Complementarity Problems

Source Problems and Definitions

Here (with minor notational changes) we repeat some material from earlier in the course.

We also follow Chapter 1 of the book

Some source problems

(a) KKT conditions of nonlinear programming

(b) Saddle problems

(c) Nash equilibrium problems

(d) Nash-Cournot production/distribution problems

(e) Economic equilibrium problems

(f) Traffic equilibrium problems

(g) Contact problems with Coulomb friction

(h) Nonlinear obstacle problems

(i) Pricing American options

(j) CPs in SPSD matrices
Definition. The (Finite-Dimensional) Variational Inequality Problem (VI) is to find a solution of
\[ u \in K, \quad \langle F(u), x - u \rangle \geq 0 \quad \text{for all } x \in K \]
where \( F : \mathbb{R}^n \to \mathbb{R}^n \) and \( K \) is a subset of \( \mathbb{R}^n \).
For given \( K \) and \( F \), the corresponding VI problem is denoted \( \text{VI}(K, F) \).
The usual assumptions imposed on \( \text{VI}(K, F) \) are that
- \( K \) is closed (and sometimes convex) and
- \( F \) is continuous (on an open set containing \( K \))

These properties facilitate theoretical results and convergence arguments.
The normal cone and generalized equation

**Definition.** Let $K$ be a closed set and let $x' \in K$. The *normal cone* at $x'$ is the set

$$\mathcal{N}(x'; K) = \{ d \in \mathbb{R}^n : d^T(y-x') \leq 0, \text{ for all } y \in K \}.$$ 

From the definition of VI $(K, F)$, it follows that $\bar{x} \in K$ solves VI $(K, F)$ if and only if $-F(\bar{x}) \in \mathcal{N}(\bar{x}; K)$.

An equivalent way to express this is to write

$$0 \in F(\bar{x}) + \mathcal{N}(\bar{x}; K).$$

Such a relation is called a *generalized equation*. 

Definition. For a cone $K \subset \mathbb{R}^n$ and a mapping $F : K \rightarrow \mathbb{R}^n$ the Complementarity Problem (CP) is to find a solution of the conditions 

$$x \in K, \quad F(x) \in K^*, \quad \text{and} \quad x^T F(x) = 0.$$ 

Since the CP is specified by $K$ and $F$, it is often denoted CP($K, F'$).

This formulation covers both the standard LCP and its generalization, the Nonlinear Complementarity Problem or (NCP) which is usually defined relative to $\mathbb{R}^n_+$ and a nonlinear, i.e., non-affine, mapping $F$.

Under these assumptions, we simply denote the problem as NCP($F'$).

If $x$ is to be solution of NCP($F'$), we must have 

$$x_i \geq 0, \quad F_i(x) \geq 0, \quad \text{and} \quad x_i F_i(x) = 0 \quad \forall i = 1, \ldots, n.$$
The complementarity problem is a special case of the variational inequality problem. In particular, we have the following result.

**Proposition.** If $K$ is a cone, the problems $CP(K, F)$ and $VI(K, F)$ have the same solutions (if any).

**Proof.** Let $\bar{x} \in SOL\ VI(K, F)$. Then we have

$$\bar{x} \in K \quad \text{and} \quad (x - \bar{x})^T F(\bar{x}) \geq 0 \quad \text{for all} \quad x \in K.$$ 

The fact that $K$ is a cone has two important implications:

$$0 \in K \implies \bar{x}^T F(\bar{x}) \leq 0 \quad \text{and} \quad 2\bar{x} \in K \implies \bar{x}^T F(\bar{x}) \geq 0.$$ 

Hence $\bar{x}^T F(\bar{x}) = 0$ and $x^T F(\bar{x}) = (x - \bar{x})^T F(\bar{x}) \geq 0$ for all $x \in K$, so that $F(\bar{x}) \in K^*$. Therefore $\bar{x} \in SOL\ CP(K, F)$.

Conversely, if $\bar{x} \in SOL\ CP(K, F)$ we have $\bar{x} \in K$ and $x^T F(\bar{x}) \geq 0$ for all $x \in K$. Since $\bar{x}^T F(\bar{x}) = 0$, we obtain

$$(x - \bar{x})^T F(\bar{x}) = x^T F(\bar{x}) \geq 0 \quad \text{for all} \quad x \in K.$$ 

Therefore $\bar{x} \in SOL\ VI(K, F)$. 
Affine problems

When $K$ is a cone and $F$ is an affine transformation from $\mathbb{R}^n$ into itself, the corresponding VI problem is called an affine variational inequality (AVI) problem. In particular, if $F(x) = q + Mx$, the problem is denoted $\text{AVI}(K, q, M)$.

Of course, when $K = \mathbb{R}_+^n$, we recover the familiar LCP $(q, M)$. 
Affine problems

When $K$ is a cone and $F$ is an affine transformation from $R^n$ into itself, the corresponding VI problem is called an affine variational inequality (AVI) problem. In particular, if $F(x) = q + Mx$, the problem is denoted $\text{AVI}(K, q, M)$.

Of course, when $K = R^n_{+}$, we recover the familiar LCP $(q, M)$.

Box constrained problems

Unlike the CP which requires a cone for the domain of its mapping, the VI problem can be defined with reference to box constraints. In such cases

$$K = \{ x \in R^n : a_i \leq x_i \leq b_i, \text{ for all } i = 1, \ldots, n \}$$

where the $a_i$ and $b_i$ all satisfy

$$-\infty \leq a_i < b_i \leq \infty.$$
Mixed complementarity problems

As we know from nonlinear programming, the Lagrange multipliers associated with equality constraints are free (not sign restricted). When we consider the KKT conditions of such problems, we obtain a different sort of complementarity problem.

**Definition.** Let $G$ and $H$ be mappings from $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}_+$ into $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}_+$, respectively. The corresponding **Mixed Complementarity Problem** (MiCP) is that of finding $(u, v) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}_+$ such that

\[
G(u, v) = 0 \quad (u \text{ free})
\]
\[
H(u, v) \geq 0 \quad v \geq 0 \quad v^T H(u, v) = 0
\]

With $G$ and $H$ given, this problem is denoted MiCP $(G, H)$.

The so-called **mixed linear complementarity problem** (MLCP) is a special case of the above.
Optimization problems and the integrability issue

Consider the abstract optimization problem

\[
(P) \quad \begin{align*}
\text{minimize} & \quad \theta(x) \\
\text{subject to} & \quad x \in K
\end{align*}
\]

where $K$ is a closed convex set, $\theta$ is a continuously differentiable function on an open set $U \subseteq \mathbb{R}^n$ containing $K$. 
Optimization problems and the integrability issue

Consider the abstract optimization problem

\[
(P) \quad \begin{align*}
\text{minimize} & \quad \theta(x) \\
\text{subject to} & \quad x \in K
\end{align*}
\]

where \( K \) is a closed convex set, \( \theta \) is a continuously differentiable function on an open set \( U \subseteq \mathbb{R}^n \) containing \( K \).

It is a general fact, called the \textit{minimum principle}, that if \( \bar{x} \) is a local minimizer of \( \theta \) on \( K \), then

\[
(x - \bar{x})^T \nabla \theta(\bar{x}) \geq 0 \quad \text{for all } x \in K.
\]

A solution of the VI \((K, \nabla \theta)\) is called a \textit{stationary point} of \((P)\), and the VI itself is called the \textit{stationary point problem} for \((P)\).

Note that when \( \theta \) is convex, a solution of the stationary point problem is a globally optimal solution of \((P)\).
Question:

Suppose $K$ is a convex set and $F$ is continuously differentiable mapping on an open set $U$ containing $K$.

When is a solution of the VI $(K, F)$ also a solution of an optimization problem having $K$ as its feasible region?

This question has to do with the “integrability” of $F$.

Definition. The continuously differentiable mapping $F : U \to \mathbb{R}^n$ is integrable if for all $x$ and $y$ in $U$, the line integral of $F$ from $x$ to $y$ is independent of any piecewise smooth path in $U$ connecting $x$ and $y$. 
Theorem. Let $U$ be an open set in $\mathbb{R}^n$ and let $F : U \to \mathbb{R}^n$ be continuously differentiable. The following statements are equivalent:

(i) There exists a function $\theta : U \to \mathbb{R}$ such that $F(x) = \nabla \theta(x)$ for all $x \in U$.
(ii) The Jacobian matrix $\nabla F(x)$ is symmetric for all $x \in U$.
(iii) The mapping $F$ is integrable on $U$.

When any of these conditions holds, the scalar function $\theta$ is given by

$$\theta(x) = \int_0^1 F(x^0 + t(x - x^0))^T(x - x^0) dt$$

where $x^0$ is an arbitrary element of $U$.

For a proof of this theorem, see J.M. Ortega and W.C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, p. 95. They attribute the result to M. Kerner.

(a) KKT conditions of mathematical programming

Consider the optimization (P) above where

\[ K = \{ x \in \mathbb{R}^n : h(x) = 0, \ g(x) \leq 0 \}. \]

Assume that \( h : \mathbb{R}^n \to \mathbb{R}^\ell \) and \( g : \mathbb{R}^n \to \mathbb{R}^m \) are differentiable.

If \( \bar{x} \) is a local minimum of \( \theta \) and a constraint qualification is satisfied, then there will exist vectors of multipliers \( \mu \) and \( \lambda \) such that

\[
\nabla \theta(\bar{x}) + \sum_{j=1}^{\ell} \mu_j \nabla h_j(\bar{x}) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{x}) = 0
\]

\[ h(\bar{x}) = 0 \]

\[ g(\bar{x}) \leq 0 \]

\[ \lambda \geq 0 \]

\[ \lambda^T g(\bar{x}) = 0 \]
**Definition.** Let $K$ be a closed set in $R^n$. The tangent cone at an point $x \in K$ is the set $T(x; K)$ of all $d \in R^n$ for which there exists a sequence of vectors $\{y^\nu\} \subset K$ and a sequence of positive scalars $\{\tau^\nu\}$ such that

$$\lim_{\nu \to \infty} y^\nu = x; \quad \lim_{\nu \to \infty} \tau^\nu = 0; \quad \text{and} \quad \lim_{\nu \to \infty} \frac{y^\nu - x}{\tau^\nu} = d.$$ 

**Remark.** If $K$ is closed and convex, and $x, y \in K$ are arbitrary, then $d = y - x \in T(x; K)$.

To see this, note that we can restrict our attention to $\tau^\nu \in (0, 1)$. For such $\tau^\nu$, define $y^\nu = x + \tau^\nu(y - x) = \tau^\nu y + (1 - \tau^\nu)x$. Clearly the $y^\nu \in K$ and $y^\nu \to x$ as $\tau^\nu \to 0$. But

$$d = (y - x) = \lim_{\nu \to \infty} \frac{y^\nu - x}{\tau^\nu}.$$
It can be shown that if $x$ is a local minimizer of the problem (P) above, then

\[(y - x)^T \nabla(x) \geq 0, \quad \text{for all } y \in x + T(x; K).\]

This is an instance of a quasi-variational problem (QVI). Note here the set by which a variational inequality is defined varies with the variable.

**Definition.** Let $K$ be a point-to-set mapping from $\mathbb{R}^n$ to subsets of $\mathbb{R}^n$. [It could happen that for some $x \in \mathbb{R}^n$, the set $K(x)$ is empty.] Let $F$ be a point-to-point mapping from $\mathbb{R}^n$ to itself. Then QVI $(K, F)$ is the problem of finding a vector $x \in K(x)$ such that

\[(y - x)^T F(x) \geq 0, \quad \forall y \in K(x).\]

When $F = 0$, the inequality becomes trivial, and the problem reduces to that of finding a fixed point of the point-to-set mapping $K$. 
**Definition.** Relative to the set $K$ defined above, *linearization cone* at $x \in K$ is the set

$$L(x; K) = \{ v \in \mathbb{R}^n : v^T \nabla h_j(x) = 0, \forall j = 1, \ldots, \ell $$

$$v^T \nabla g_i(x) \leq 0, \forall i \in A(x) \}$$

where $A(x) = \{ i : g_i(x) = 0 \}$ is called the *active set* at $x$.

*Abadie’s constraint qualification* is: $T(x; K) = L(x; K)$. 
Note that $\mathcal{L}(x; K)$ is a polyhedral cone, whereas, in general, $\mathcal{T}(x; K)$ is a cone but is not necessarily convex.

When Abadie’s constraint qualification holds, it follows that

$$
\mathcal{T}(x; K)^* = \mathcal{L}(x; K)^* = \{ v \in \mathbb{R}^n : \exists (\mu, \lambda_{\mathcal{A}(x)}) \text{ with } \lambda_{\mathcal{A}(x)} \geq 0 \text{ such that } \\
v + \sum_{j=1}^{\ell} \mu_j \nabla h_j(x) + \sum_{i \in \mathcal{A}(x)} \lambda_i \nabla g(x) = 0 \}
$$
**Proposition.** If $K \subset \mathbb{R}^n$ is convex and $x \in K$ is arbitrary, then

$$\mathcal{T}(x; K)^* = -\mathcal{N}(x; K).$$

**Proof.** Let $d \in \mathcal{T}(x; K)^*$ and $y \in K$ be arbitrary. Since $K$ is convex, $y - x$ is a tangent vector to $K$ at $x$. Hence $0 \leq d^T(y - x)$. This means $-d \in \mathcal{N}(x; K)$. But this is the same as saying $d \in -\mathcal{N}(x; K)$.

Conversely, suppose $d \in -\mathcal{N}(x; K)$. Then $-d \in \mathcal{N}(x; K)$, so $-d^T(y - x) \leq 0$ for all $y \in K$. So we have $-d^T(y - x) \leq 0$ for all $y \in K$. From this it follows that $d$ makes an acute angle with every element of $\mathcal{T}(x; K)$. In other words, $d \in \mathcal{T}(x; K)^*$.

Thus, when $K$ is convex, we have a polyhedral representation of $\mathcal{N}(x; K)$.
When $K$ is convex and Abadie’s constraint qualification holds at $x$, we obtain the KKT system (conditions)

\[
\nabla \theta(x) + \sum_{j=1}^{\ell} \mu_j \nabla h_j(x) + \sum_{i=1}^{m} \lambda_i \nabla g(x) = 0
\]

\[
h(x) = 0
\]

\[
g(x) \leq 0
\]

\[
\lambda \geq 0
\]

\[
\lambda^T g(x) = 0
\]