Pivoting Methods for the LCP

Preliminaries

Let \((q, M)\) be of order \(n\). We look for certain sign configurations in the data:

1. \(q \geq 0 \implies z = 0 \in \text{SOL}(q, M)\).

2. \(q_r < 0\) and \(m_{rj} \leq 0 \ \forall \ j \implies \text{FEA}(q, M) = \emptyset \implies \text{SOL}(q, M) = \emptyset\)

Either of these would give reason to stop computing. We call them terminal sign configurations.

Later we'll see another terminal sign configuration.
There exist feasible LCPs for which no complementary feasible solution exists.

**Example.** Let $M$ have the sign pattern $\begin{bmatrix} + & - \\ + & - \end{bmatrix}$ and let $q$ have the sign pattern $\begin{bmatrix} + \\ - \end{bmatrix}$.

In any such case $q \in \text{pos}[I, -M]$ so $(q, M)$ is feasible, but if $Me = 0$, then $K(M)$ is nonconvex. This implies $M \notin Q_0$. The problem is feasible but has no solution.

**Definition.** An LCP algorithm that either finds a solution or indicates that no solution exists is said to *process* the problem.

Note that we use the word “solution” in a restrictive sense: a vector $z$ is a solution of an LCP if and only if it is feasible and complementary.
The purpose of pivoting in an LCP is to make the detection of terminal sign configurations easy.

This is somewhat related to an interpretation of the simplex method for linear programming. Consider the LP

$$\text{minimize } x_0 = c^T x \text{ subject to } Ax = b, \ x \geq 0 \quad (A \in R^{m \times n})$$

in canonical form with respect to a feasible basis $[A_{j_1}, \ldots, A_{j_m}]$. For ease of discussion, assume $j_i = n - m + i$.

The next panel shows tableaus with terminal sign configurations.
\[
\begin{array}{cccccc|c}
\mathbf{x}_0 & \mathbf{x}_1 & \cdots & \mathbf{x}_{n-m} & \mathbf{x}_{n-m+1} & \cdots & \mathbf{x}_n & 1 \\
1 & * & \cdots & * & 0 & \cdots & 0 & * \\
0 & * & \cdots & * & 1 & \cdots & 0 & \oplus \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & * & \cdots & * & 0 & \cdots & 1 & \oplus
\end{array}
\]

\((* = \text{element of any sign})\)

\[
\begin{array}{cccccc|c}
\mathbf{x}_0 & \mathbf{x}_1 & \cdots & \mathbf{x}_{n-m} & \mathbf{x}_{n-m+1} & \cdots & \mathbf{x}_n & 1 \\
1 & \ominus & \cdots & \ominus & 0 & \cdots & 0 & * \\
0 & * & \cdots & * & 1 & \cdots & 0 & \oplus \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & * & \cdots & * & 0 & \cdots & 1 & \oplus
\end{array}
\]

(basic solution optimal)

\[
\begin{array}{cccccc|c}
\mathbf{x}_0 & \mathbf{x}_1 & \cdots & \mathbf{x}_{n-m-1} & \mathbf{x}_{n-m} & \mathbf{x}_{n-m+1} & \cdots & \mathbf{x}_n & 1 \\
1 & * & \cdots & * & + & 0 & \cdots & 0 & * \\
0 & * & \cdots & * & \ominus & 1 & \cdots & 0 & \oplus \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & * & \cdots & * & \ominus & 0 & \cdots & 1 & \oplus
\end{array}
\]

\((x_0 \rightarrow -\infty)\)
For many kinds of matrices $M \in \mathbb{R}^{n \times n}$, solving an LCP $(q, M)$ is a matter of finding a nonsingular principal submatrix $M_{\alpha\alpha}$ of $M$ such that $q$ belongs to the complementary cone $\text{pos} \, C_M(\alpha)$.

When this is the case, performing a principal pivot on $M_{\alpha\alpha}$ will reveal the terminal sign configuration

$$\bar{q} \geq 0 \quad \text{where} \quad \bar{q}_\alpha = -M_{\alpha\alpha}^{-1}q_\alpha$$

$$\bar{q}_{\bar{\alpha}} = q_{\bar{\alpha}} - M_{\bar{\alpha}\alpha} M_{\alpha\alpha}^{-1}q_\alpha$$

The trick is to identify the index set $\alpha$. Ordinarily, this is done through a sequence of *simple pivots*. 
This approach is facilitated by matrix properties that make it possible to carry out a sequence of pivots in such a way that a terminal sign configuration is reached after finitely many steps—and, one would hope, not too many of them.

Not all matrix classes are candidates for the application of this idea. Perhaps the largest (known) class for which (simple or block) principal pivoting will work in this way is RSU.

We will begin by considering two of its subclasses, P and PSD.
Invariance theorems

The following is due to A.W. Tucker (1963).

**Theorem.** If $M \in \mathbf{P}$ and $M' = \varphi_\alpha(M)$ is a principal pivotal transform of $M$, then $M' \in \mathbf{P}$.

**Proof.** Assume $M \in R^{n \times n}$. Let $\beta \subseteq \{1, \ldots, n\}$ be arbitrary. By Schur’s determinantal formula (see slides #8), we have

$$
\det M'_{\beta\beta} = \frac{\det M_{\gamma\gamma}}{\det M_{\alpha\alpha}}
$$

where $\gamma = \alpha \Delta \beta$ (the symmetric difference of $\alpha$ and $\beta$).

A stronger form of this theorem is due to T.D. Parsons (1970).

**Theorem.** $M \in \mathbf{P}$ if and only if every principal pivotal transform of $M$ has a positive diagonal.
Theorem. If $M \in \text{PSD}$ and $M' = \wp_\alpha(M)$ is a principal pivotal transform of $M$, then $M' \in \text{PSD}$.

Proof. Let $M_{\alpha\alpha}$ be a nonsingular principal submatrix of $M$ and write

$$w_\alpha = M_{\alpha\alpha}z_\alpha + M_{\alpha\bar{\alpha}}z_{\bar{\alpha}}$$
$$w_{\bar{\alpha}} = M_{\bar{\alpha}\alpha}z_\alpha + M_{\bar{\alpha}\bar{\alpha}}z_{\bar{\alpha}}$$

Then we have $z^TMz = z_\alpha^Tw_\alpha + z_{\bar{\alpha}}^Tw_{\bar{\alpha}} \geq 0$. Now let $M' = \wp_\alpha(M)$.

Suppose we write

$$w_\alpha = M'_{\alpha\alpha}z_\alpha + M'_{\alpha\bar{\alpha}}z_{\bar{\alpha}}$$
$$w_{\bar{\alpha}} = M'_{\bar{\alpha}\alpha}z_\alpha + M'_{\bar{\alpha}\bar{\alpha}}z_{\bar{\alpha}}$$

Then we have

$$z_\alpha = M_{\alpha\alpha}w_\alpha + M_{\alpha\bar{\alpha}}z_{\bar{\alpha}}$$
$$w_{\bar{\alpha}} = M_{\bar{\alpha}\alpha}w_\alpha + M_{\bar{\alpha}\bar{\alpha}}z_{\bar{\alpha}}$$
Thus we can write

\[ z^T M' z = z_{\alpha}^T w_\alpha + z_{\bar{\alpha}}^T w_{\bar{\alpha}} \]

\[ = w_\alpha^T z_\alpha + z_{\bar{\alpha}}^T w_{\bar{\alpha}} = \begin{bmatrix} w_\alpha \\ z_{\bar{\alpha}} \end{bmatrix}^T M \begin{bmatrix} w_\alpha \\ z_{\bar{\alpha}} \end{bmatrix} \geq 0. \]

Hence \( M' \in \text{PSD}. \)
Thus we can write
\[
z^T M' z = z^T \alpha w_{\alpha} + z^T \bar{\alpha} w_{\bar{\alpha}} = w_{\alpha}^T z_{\alpha} + z_{\bar{\alpha}}^T w_{\bar{\alpha}} = \begin{bmatrix} w_{\alpha} \\ z_{\bar{\alpha}} \end{bmatrix}^T M \begin{bmatrix} w_{\alpha} \\ z_{\bar{\alpha}} \end{bmatrix} \geq 0.
\]
Hence \( M' \in \text{PSD}. \)

Using a slight refinement of this proof, we can establish

**Theorem.** If \( M \in \text{PD} \) and \( M' = \varphi_{\alpha}(M) \) is a principal pivotal transform of \( M \), then \( M' \in \text{PD}. \)

Thus, \( \text{P} \), \( \text{PSD} \), and \( \text{PD} \) are invariant under principal pivoting.
Simple Principal Pivoting Methods for the LCP

These algorithms use simple diagonal pivots only.

We need a bit of notation to keep track of things.

The fundamental equation for the problem \((q, M)\) is \(w = q + Mz\).

We attach superscripts \(\nu = 0, 1, \ldots\) to the variables and the data and then write the original system as

\[
w^0 = q^0 + M_0 z^0.
\]

But we also use \(w^\nu (z^\nu)\) to denote the variables that are basic (nonbasic) after \(\nu\) iterations. The corresponding system is

\[
w^\nu = q^\nu + M^\nu z^\nu.
\]

For all \(\nu = 0, 1, \ldots\) we have

\[
\{w^\nu_i, z^\nu_i\} = \{w_i, z_i\} \quad \text{for all } i.
\]
Bard-type methods

In G. Zoutendijk’s classic book (and Ph.D. thesis) *Methods of Feasible Directions* (1960) there is a direction-finding problem (QP) which boils down to the LCP \((Pb, PP^T)\). The same ideas were also studied in the early sixties by Y. Bard (1972).

These are probably called Bard-type methods because “Bard” is so much easier to spell than “Zoutendijk.”

Their algorithms apply to problems of the form \((Pb, PP^T)\) and use only simple principal pivots. Note that \(M = PP^T\) is symmetric and positive semidefinite. The vector \(q\) is in the column space of \(P\).

Before getting to the algorithm, we state and prove a theorem that concerns a slight generalization of the problem \((Pb, PP^T)\).
Theorem. Let \((q, M)\) be an LCP with \(M = PAP^T\) and \(q = Pb\) where \(A\) is positive definite. Let \((q', M') = \varphi_\alpha(q, M)\) be obtained by a (possibly vacuous) principal (block) pivot on the principal submatrix \(M_{\alpha\alpha}\) of \(M\). Then \(q'_s \neq 0\) only if \(m'_{ss} > 0\).

Proof. Note that \(M_{\alpha\alpha} = P_\alpha A(P_\alpha)^T\) is nonsingular, so the rows of \(P_\alpha\) are linearly independent. Indeed, \(M_{\alpha\alpha} \in \text{PD}\) and \(m'_{ss} > 0\) for all \(s \in \alpha\).

Since \(M' \in \text{PSD}\), we have \(m'_{ss} \geq 0\) for all \(s \in \bar{\alpha}\). Suppose \(m'_{ss} = 0\). Put \(\alpha' = \alpha \cup \{s\}\). Note that \(M_{\alpha'\alpha'}\) must be singular since

\[
0 = m'_{ss} = \det M_{\alpha'\alpha'}/\det M_{\alpha\alpha}.
\]

Hence \(P_s = v_\alpha^T P_\alpha\) for some \(v_\alpha\), and \(P_s A(P_\alpha)^T = v_\alpha^T M_{\alpha\alpha}\). Now,

\[
q'_s = P_s b - P_s A(P_\alpha)^T (M_{\alpha\alpha})^{-1} P_\alpha b
= v_\alpha^T P_\alpha b - v_\alpha^T P_\alpha A(P_\alpha)^T (M_{\alpha\alpha})^{-1} P_\alpha b = 0.
\]
This theorem implies that we can depend on a diagonal entry of $M^\nu$ to be positive when we need it as a pivot element.

**Algorithm (Zoutendijk/Bard)**

**Step 0. Initialize.** Input either $(q^0, M^0) = (Pb, PAP^T)$ with $A \in \mathbf{PD}$ or $(q^0, M^0)$ with $M^0 \in \mathbf{P}$. Set $\nu = 0$.

**Step 1. Test for termination.** If $q^\nu \geq 0$, stop. $z^\nu = 0$ solves $(q^\nu, M^\nu)$. Recover solution of $(q^0, M^0)$.

**Step 2. Choose pivot row.** Choose an index $r$ such that $q^\nu_r < 0$.

**Step 3. Pivot.** Pivot on $m^\nu_{rr}$. (Execute $\langle w^\nu_r, z^\nu_r \rangle$.)

\[
\begin{align*}
w^\nu_{r+1} &= z^\nu_r & z^\nu_{r+1} &= w^\nu_r \\
 w^\nu_{i+1} &= w^\nu_i & z^\nu_{i+1} &= z^\nu_i & i \neq r
\end{align*}
\]

Return to Step 1 with $\nu \leftarrow \nu + 1$. 

Issues As stated, this algorithm might *cycle*, that is, perform a sequence of pivot steps that returns to a previously encountered complementary basis.

Example (L. Watson). Consider the LCP \( q, M \) where

\[
q = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 1.0 & 0.2 \\ 0.2 & 1.0 \\ 0.0 & 0.2 & 0.1 \end{bmatrix}.
\]

The matrix \( M \) here belongs to \( \textbf{P} \), so the diagonal elements of every principal pivotal transform of \( M \) are positive. Hence all simple principal pivots are available.

If we carry out the pivots

\[
\langle w_1, z_1 \rangle, \langle w_3, z_3 \rangle, \langle z_1, w_1 \rangle, \langle w_2, z_2 \rangle, \langle z_3, w_3 \rangle, \langle w_1, z_1 \rangle, \langle z_2, w_2 \rangle
\]

we return to the second tableau. This is an instance of *cycling*. 
There is nothing done in the Zoutendijk/Bard algorithm the prevent cycling, though at least in the Zoutendijk problem there is a way to prevent cycling and thereby to assure finiteness. To show that the algorithm really solves \((Pb, PAP^T)\), Zoutendijk introduced a lexicographic approach.

One can write down a tableau of the form

\[
\begin{array}{c|cc|c}
1 & z & x \\
q & M & B \\
\end{array}
\]

where \(B\) has lexicographically positive and linearly independent rows.

The selection of the pivot row \(r\) is done according to the rule

\[
b^\nu = \frac{B_r}{q_r^\nu} = \text{lexicomax} \left\{ \frac{B_i^\nu}{q_i^\nu} : q_i^\nu < 0 \right\}
\]

It can be shown that the sequence of vectors \(b^\nu\) will decrease strictly in the lexicographic sense. This will prevent the repetition of a basis, and since there are only finitely many bases to begin with, the method will be finite.
Murty’s least-index rule

Step 0. Initialize. Input \((q^0, M^0)\) with \(M^0 \in \mathbf{P}\). Set \(\nu = 0\).

Step 1. Test for termination. If \(q^\nu \geq 0\), stop. \(z^\nu = 0\) solves \((q^\nu, M^\nu)\). Recover solution of \((q^0, M^0)\).

Step 2. Choose pivot row. Choose \(r\) so that \(r = \min \{i : q^\nu_i < 0\}\).

Step 3. Pivot. Pivot on \(m^\nu_{rr}\). (Execute \(\langle w^\nu_r, z^\nu_r \rangle\).)

\[
\begin{align*}
w^{\nu+1}_r &= z^\nu_r & z^{\nu+1}_r &= w^\nu_r \\
w^{\nu+1}_i &= w^\nu_i & z^{\nu+1}_i &= z^\nu_i & i \neq r
\end{align*}
\]

Return to Step 1 with \(\nu \leftarrow \nu + 1\).

Note: \(r\) is chosen as the least index; it need not be true that

\[
q^\nu_r = \min \{q^\nu_i : q^\nu_i < 0\}.
\]
It can be shown that Murty’s least-index method will solve the LCP $(q, M)$ when $M \in \mathbf{P}$.

It will do so without cycling.

A notable feature of this algorithm is that it does nothing to preserve the nonnegativity of basic variables.

This eliminates the computational effort associated with preventing nonnegative basic variables from becoming negative.

The next algorithm will not have this feature.
Here we are going to discuss a method of processing LCP’s with symmetric positive semidefinite matrices.

We will work with principal pivotal transforms of the fundamental system $w^0 = q^0 + M^0 z^0$. The pivots will also be simple principal pivots.

To do this, we need some new terminology. Suppose $w^\nu = q^\nu + M^\nu z^\nu$ is a principal pivotal transform of the original fundamental system. Let $(w^\nu, z^\nu)$ be a basic solution of this system. Then $z^\nu = 0$.

If we fix attention on a specific basic variable $w^\nu_r$ whose current value is $q^\nu_r < 0$ and we are attempting to make it become 0, we call it the current distinguished basic variable.
Since $m_{rr}^{\nu} \geq 0$, we can attempt to make the distinguished variable increase by increasing its complement $z_{rr}^{\nu}$ which in this role is called the driving variable. We are going to try to drive the distinguished variable up to zero.

If $m_{rr}^{\nu} > 0$, then $w_{r}^{\nu}$ would reach 0 when $z_{r}^{\nu} = -q_{r}^{\nu}/m_{rr}^{\nu} > 0$ in which case the increase of $z_{r}^{\nu}$ would be blocked. Going farther would make $w_{r}^{\nu}z_{r}^{\nu} > 0$.

Let $\alpha = \alpha(\nu)$ be the set of all $i \in \{1, \ldots, n\}$ such that $z_{i} = z_{i}^{0}$ is one of the basic variables at stage $\nu$. That is, $z_{i}$ is one of the components of $w^{\nu}$. In fact, $w_{\alpha}^{\nu}$ will be composed of $z$-variables.

Let $\beta = \bar{\alpha}$, the complementary index set. Then $w_{\beta}^{\nu}$ will be composed of $w$-variables.

The composition of the sets $\alpha$ and $\beta$ will change from iteration to iteration.
Processing symmetric monotone LCP's

Algorithm (Dantzig; van de Panne & Whinston)

**Step 0. Initialize.** Input \((q, M) = (q^0, M^0)\) with \(M = M^T \in \text{PSD}\).
Set \(\nu = 0, \alpha = \emptyset, \beta = \{1, \ldots, n\}\).

**Step 1. Test for termination.** Break ties arbitrarily. Let
\[
\nu \in \arg \min \{q^\nu_i : i \in \beta\}.
\]

**Step 1A.** If \(q^\nu_r \geq 0\), stop: \(\bar{z}_\alpha = q^\nu_\alpha\) and \(\bar{z}_\beta = 0\) gives a solution.

**Step 1B.** If \(q^\nu_r < 0\), choose \(w^\nu_r\) as the distinguished variable and \(z^\nu_r\) as the driving variable.

**Step 1C.** If \(m^\nu_{rr} = 0\) and \(m^\nu_{ir} \geq 0\) for all \(i\), stop: there is no solution.
Step 2. Determine blocking variable. Variables eligible to block $z_r^\nu$ are the $w_i^\nu$ such that $i \in \alpha \cup \{r\}$. Use the minimum ratio test to determine the blocking variable. Break ties arbitrarily, but choose $w_r^\nu$ as blocking variable if it is involved in a tie for the first eligible basic variable to reach 0.

Step 3. Pivoting. Let $w_s^\nu$ be the basic variable that blocks $z_r^\nu$. Pivot $\langle w_s^\nu, z_s^\nu \rangle$ and put

\[
\begin{align*}
w_s^{\nu+1} &= z_s^{\nu} & z_s^{\nu+1} &= w_s^{\nu} \\
w_i^{\nu+1} &= w_i^{\nu} & z_i^{\nu+1} &= z_i^{\nu} & i \neq s
\end{align*}
\]

If $s = r$, $\alpha \leftarrow \alpha \cup \{r\}$ and $\beta \leftarrow \beta \setminus \{r\}$.
If $s \neq r$, $\alpha \leftarrow \alpha \setminus \{r\}$ and $\beta \leftarrow \beta \cup \{r\}$.
Return to Step 1 with $\nu \leftarrow \nu + 1$. 
Remark. As we know, a symmetric monotone LCP \((q, M)\) is equivalent (has the same solutions, if any) as the convex QP

\[
\text{minimize } q^T z + \frac{1}{2} z^T M z \quad \text{subject to } z \geq 0.
\]

It is easy to prove that for all \(z\) if \(z^T (q + M z) = z^T w = 0\), then

\[
f(z) := q^T z + \frac{1}{2} z^T M z = \frac{1}{2} q^T z.
\]

So let \(\theta := 2f(z)\). Then for all \(z\) such that \(z^T w = 0\), we have \(\theta = q^T z\).

We can use this idea to create the tabular form

\[
\begin{array}{ccc}
1 & z \\
\theta & 0 & q^T \\
w & q & M \\
\end{array}
\]

What happens when we perform a pivot on the block \(M_{aa} \in \text{PD}\)?
We get

<table>
<thead>
<tr>
<th></th>
<th>$1$</th>
<th>$w_\alpha$</th>
<th>$z_\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>$-q_\alpha^T M_{\alpha\alpha}^{-1} q_\alpha$</td>
<td>$q_\alpha^T M_{\alpha\alpha}^{-1}$</td>
<td>$q_\beta^T - q_\alpha^T M_{\alpha\alpha}^{-1} M_{\alpha\beta}$</td>
</tr>
<tr>
<td>$z_\alpha$</td>
<td>$-M_{\alpha\alpha}^{-1} q_\alpha$</td>
<td>$M_{\alpha\alpha}^{-1}$</td>
<td>$-M_{\alpha\alpha}^{-1} M_{\alpha\beta}$</td>
</tr>
<tr>
<td>$w_\beta$</td>
<td>$q_\beta - M_{\beta\alpha} M_{\alpha\alpha}^{-1} q_\alpha$</td>
<td>$M_{\beta\alpha} M_{\alpha\alpha}^{-1}$</td>
<td>$M_{\beta\beta} - M_{\beta\alpha} M_{\alpha\alpha}^{-1} M_{\alpha\beta}$</td>
</tr>
</tbody>
</table>