1. Problems 28.5 and 28.7 of Vazirani.
2. Exercise 2.1 from Lovasz’s Survey.
3. Prove that a connected graph $G$ with maximum eigenvalue $\lambda_1$ is bipartite if and only if $-\lambda_1$ is also an eigenvalue.
4. In this exercise, we will prove Polya’s Theorem using the connections between random walks on a graph and electric currents in a corresponding network of resistors.

**Polya’s Theorem.** Simple random walk on a $d$-dimensional lattice is recurrent for $d = 1, 2$ and transient for $d > 2$.

Let $G$ denote the lattice graph. Let $G(r)$ be the subgraph of $G$ induced by the nodes with distance at most $r$ from the origin. Denote by $\partial G(r)$ the set of nodes that are exactly $r$ units away from the origin. Let $p(r)$ be the probability that a random walk on $G(r)$ starting at the origin reaches $\partial G(r)$ before returning to origin and let $p = \lim_{r \to \infty} p(r)$ be the probability for the infinite graph. An infinite walk is recurrent if and only if the limit is 0.

To determine $p$ electrically, we simply ground all the nodes of $\partial G(r)$, maintain the origin at one volt, and measure the current flowing into the circuit. We will have

$$p(r) = \frac{i(r)}{2d} = \frac{1}{2dR_{\text{eff}}(r)}$$

where $d$ is the dimension and $R_{\text{eff}}(r)$ is the effective resistance from the origin to $\partial G(r)$.

Let $R_{\text{eff}} = \lim_{r \to \infty} R_{\text{eff}}(r)$. Then $p = 0$ iff $R_{\text{eff}}$ is infinite.

By using the above connection, prove Polya’s theorem for one dimension.

(a) Show that simple random walk on $d$-dimensional lattice is recurrent for $d = 1$.

For $d = 2$, we need to use a shorting technique: shorting a group of nodes between the origin and outside boundary will only reduce resistance.

(b) Show that simple random walk on $d$-dimensional lattice is recurrent for $d = 2$.

To get an upper bound for higher dimensions, we need to use a cutting technique to cut the grid into trees. Compute the effective resistance of an infinite binary tree and then embed it into a 3-dimensional grid to prove the following:

(c) Show that simple random walk on $d$-dimensional lattice is transient for $d = 3$. 