Problem 1. Prove that the edges of a bipartite graph with maximum degree $\Delta$ can be colored with $\Delta$ colors such that no two edges that share a vertex have the same color.

Problem 2. A square matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is doubly stochastic if the entries of the matrix are nonnegative, and the sum of entries in every row and column is equal to one. The Birkhoff-von Neumann theorem states that one can write any doubly stochastic matrix as a convex combination of permutation matrices. Prove this theorem and show that we can write any doubly stochastic matrix as a convex combination of at most $n^2 - n$ permutation matrices.

Problem 3. We have the following Theorem:

**Theorem 1** Let $G$ be a graph and let $M$ be a matching in $G$ and let $B$ be a blossom with respect to $M$. Then $M$ is a maximum size matching in $G$ if and only if $M/B$ is a maximum size matching in $G/B$.

Give an example of a graph $G$, a matching $M$ and a blossom $B$ for $M$ such that a maximum matching $M^*$ in $G/B$ does not lead to a maximum matching in $G$. Explain why this does not contradict Theorem 1.

Problem 4. A stable set $S$ (sometimes, it is called also an independent set) in a graph $G = (V, E)$ is a set of vertices such that there are no edges between any two vertices in $S$. If we let $P$ denote the convex hull of all (incidence vectors of) stable sets of $G = (V, E)$, it is clear that $x_i + x_j \leq 1$ for any edge $(i, j) \in E$ is a valid inequality for $P$.

1. Give a graph $G$ for which $P$ is not equal to

$$
\{x \in \mathbb{R}^{|V|} : x_i + x_j \leq 1 \quad \text{for all } (i, j) \in E \}
$$

$$
x_i \geq 0 \quad \text{for all } i \in V
$$

2. Show that if the graph $G$ is bipartite then $P$ equals

$$
\{x \in \mathbb{R}^{|V|} : x_i + x_j \leq 1 \quad \text{for all } (i, j) \in E \}
$$

$$
x_i \geq 0 \quad \text{for all } i \in V
$$

Problem 5. Consider a bipartite graph $G = (V, E)$ with bipartition $(A, B)$. For $X \subseteq A$, define $\text{def}(X) = |X| - |N(X)|$ where $N(X) = \{b \in B : \exists a \in X \text{ with } (a, b) \in E\}$. Let

$$
\text{def}_{\text{max}} = \max_{X \subseteq A} \text{def}(X)
$$

Since $\text{def}(\emptyset) = 0$, we have $\text{def}_{\text{max}} \geq 0$.

1. Generalize Hall’s theorem by showing that the maximum size of a matching in a bipartite graph $G$ equals $|A| - \text{def}_{\text{max}}$.  

2. For any 2 subsets $X, Y \subseteq A$, show that
\[
def(X \cup Y) + \def(X \cap Y) \geq \def(X) + \def(Y)
\]

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**Problem 6.** Derive the Hall’s marriage theorem from Tutte’s theorem.

**Problem 7. (extra credit)** A graph $G = (V, E)$ is said to be *factor-critical* if, for all $v \in V$, we have that $G \setminus \{v\}$ contains a perfect matching. In parts (a) and (b) below, $G$ is a factor critical graph.

1. Let $U$ be any minimizer in the Tutte-Berge formula for $G$. Prove that $U = \emptyset$.

2. Deduce that when Edmonds algorithm terminates the final graph (obtained from $G$ by shrinking flowers) must be a single vertex.

3. Given a graph $H = (V, E)$, an *ear* is a path $v_0 - v_1 - v_2 - \ldots - v_k$ whose endpoints ($v_0$ and $v_k$) are in $V$ and whose internal vertices ($v_i$ for $1 \leq i \leq k - 1$) are not in $V$. We allow that $v_0$ be equal to $v_k$, in which case the path would reduce to a cycle. Adding (a ‘trivial’ ear) simply means adding an edge to $H$. An ear is called *odd* if $k$ is odd, and even otherwise; for example, a trivial ear is odd.

   (a) Let $G$ be a graph that can be constructed by starting from an odd cycle and repeatedly adding odd ears. Prove that $G$ is factor-critical.

   (b) Prove the converse that any factor-critical graph can be build by starting from an odd cycle and repeatedly adding odd ears.

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**Problem 8. (Project Definition)** Define your project title and team members. Projects should be done in teams of maximum two people.