Prophet Inequalities for Matching

Recap

Online bipartite matching (OBM)
- Fractional: $1 - \frac{1}{e}$, tight
- Integral: $1 - \frac{1}{e}$, tight
- Adwords (small bids): $1 - \frac{1}{e}$, tight

All adversarial.

Today

OBM in a Bayesian environment.

Edge-Arrival

- Bipartite graph $G = (A \cup B, E)$ given upfront along with ordering $\preceq$ of $E$
- $E$ is known
- $\mathcal{E}$
- Edge $e$ has weight sampled from $X_e \sim \text{exp}(\lambda)$ (all indep.)
- Edges arrive in order $E_0(1), E_0(2), \ldots, E_0(n)$
- Reveals weight $w_e \sim X_e$, we must decide irrevocably whether to match

Good benchmark?

Today’s focus: Optimum offline algorithm, opt_off

Sees (random) realization of entire graph, gets max-weight matching.

Def Algorithm $A$ is $\alpha$-competitive if

$$E[A] \leq \alpha \cdot E[\text{opt}_\text{off}]$$
OPT off also referred to as prophet (hence the name "prophet inequalities")

**Starting Point** Star graph (i.e., single-item)

*E.g.*

\[
\text{Unif}[1, 100] \rightarrow 7 \quad \text{(reject)}
\]

\[
10 \cdot \text{Ber}(1/2) \rightarrow 0 \quad \text{(reject)}
\]

\[
\text{Unif}[4, 8] \rightarrow 6.5 \quad \text{(accept)}
\]

**Obs** Optimal algorithm computable via backwards induction.

**Obs** No online algo is \( \geq 1/2 \) - competitive.

\[
E[A] = 1
\]

\[
E[\text{opt}
\text{off}] = 6(\frac{1}{e}) + (1 - e) 1
\]

\[
= 2 - e
\]
Thm [KSG’78] There is always a $\frac{1}{2}$-competitive algorithm.

We will see a proof via Online Contention Resolution Schemes (OCRS).

To motivate this, we introduce the \textit{ex-ante} relaxation.

$g_e: [0,1] \rightarrow \mathbb{R}_{\geq 0}$

$g_e: P \mapsto \mathbb{E}[x_e \mid x_e \text{ realized in top } P \text{ quantile of its distribution}].$

\underline{Ex-ante Relaxation}

$$\max \sum_e g_e(x_e)$$

s.t. $x_e \geq 0,$ $\sum_e x_e \leq 1.$

\underline{Obs. Ex-ante-OPT \geq \mathbb{E}[opt_{off}].}

Strategy Let $x_e^*$ be optimum, conditioned on $e$ realizing in top $x_e^*$ quantile, accept w.p. $\geq \frac{1}{2}.$

[FSZ’16]

\underline{Single-item OCRS} Given $n$ items $e_1, \ldots, e_n$

where $e_i$ is active independently w.p. $x_i \geq 0,$

and $\frac{1}{\sum} x_i \leq 1.$
• Edges arrive in order and reveal their active status

• Want to design an algorithm to accept $\leq 1$ active element.

**Def** $A$ is $c$-balanced if for all $i$,
\[
P[A \text{ selects } e_i] \geq c \cdot x_i.
\]

**Claim** There exists a $1/2$-balanced single-item OCRS.

**Pr** Upon arrival of $e_i$, if no edge selected so far and $e_i$ active, choose it w.p.
\[
\frac{1/2}{1 - \frac{1}{2} \sum_{i' < i} x_{i'}}.
\]

Show $P[\text{select } e_i] = \frac{1}{2} x_i$ by induction on $i$.

**Online algo.** Let $\{x^*_e\}$ be optimum for ex-ante.
Treat $e$ as "active" iff $X_e$ realizes in top $x^*_e$-quantile.
Run $1/2$-balanced OCRS
Next Edge-arrival matching via edge-arrival OCRS.

* \( \{ x_e \} \text{ now in matching polytope} \)

Natural \( c \)-balanced algorithm

If \( e = (uv) \) is active and feasible to add, do so w.p. \( c / \mathbb{P}[u,v \text{ free}] \).

Claim [EFGT '20]. Above is well-defined with \( c = \sqrt[3]{3} \).

* Proof Union bound!

\[
\mathbb{P}[u,v \text{ free}] \geq 1 - \mathbb{P}[u \text{ matched}] + \mathbb{P}[v \text{ matched}]
\]

(by induction) \[ \Rightarrow 1 - c \left( \sum_{e' \in e} x_{e'} + \sum_{e' \in e N(u)} x_{e'} \right) \]

\[ \geq 1 - 2c \]

So \[ \frac{c}{\mathbb{P}[u,v \text{ free}]} \leq \frac{c}{1 - 2c} = 1 \]

\[ \uparrow \text{ for } c = \sqrt[3]{3} \]

Note Current bounds on comp. ratio: \( [0.341, \frac{3}{7}] \) (bipartite)
Prophet Secretary for matching

Motivation: how important is adversarial order?

Starting point Star graph (i.e., single-item) \( \{ 1, \frac{1}{\epsilon}, \text{Ber} (\epsilon) \} \) only gives hardness of \( 3/4 \)

Q Can we beat \( 1/2 \)?

A Yes!

Single-item Ro-OCKS \( \epsilon \) active w.p. \( x_\epsilon \), \( \epsilon \)'s arrive in uniformly random order, want

\[ \Pr[\text{select active } \epsilon] \geq C \cdot x_\epsilon \]

Obs \( C \leq 1 - \frac{1}{\epsilon} \)

If consider \( x_\epsilon = \frac{1}{n} \) for \( n \) els \( \epsilon \)

For that example, taking the first active elt. works. Not always, however...

say \( x_1 = \epsilon \), \( x_2 = 1 - \epsilon \)

\[ \Pr[\text{take } x_1] = \frac{1}{2} \cdot \epsilon + \frac{1}{2} \cdot (1 - \epsilon) = \epsilon \left( \frac{1}{2} + \frac{\epsilon}{2} \right) \]
Tuni [LS'18] Can achieve $c = (-Ye)$.

Pf Sample random arrival times $t_1, \ldots, t_n \sim \text{Unif}[0, 1]$ independently.

\textbf{Algorithm}

Upon arrival of $e_i$, if active take w.p. $\frac{e^{-x_i t_i}}{\text{downsampling}}$

\textbf{Analysis}

Analyze

(*) $\text{Pr}[e_i \text{ available } | t_i = Y]$

This happens iff for every $j \neq i$, $t_j > x_i$, $e_j$ inactive, or $e_j$ did not survive downsampling.

(*) = \prod_{j \neq i} \left( 1 - \int_0^{x_i} e^{-zx_j} \, dz \right)

= \prod_{j \neq i} \left( 1 - \left[ e^{-zx_j} \right]_0^{x_i} \right)

= \prod_{j \neq i} e^{-yx_j} = e^{-y \sum_{j \neq i} x_j}$
\[ P[\text{select } e_i] = \int_0^1 x_i e^{-x_i y} e^{-\sum_{j \neq i} x_j} \, dy \]

\[ \geq \int_0^1 e^{-y} \, dy = x_i \left( 1 - \frac{1}{e} \right). \]

**Discussion**

1 - \( \frac{1}{e} \) is tight for RO-OLRS, but still gaps in knowledge for prophet secretary.

Esfandiari et al. '15: \[ [1 - \frac{1}{e}, 0.75] \]

Azar et al. '18: \[ \geq 1 - \frac{1}{e} + \frac{\sqrt{400}}{400} \]

Correa et al. '21: \[ [0.669, 0.732] \]

**Beyond star-graphs**

**Claim** Can achieve a 0.432-approx. for matching in general graphs.

**Proof sketch** Apply same downsampling as for single-item!