Optimal Allocation: Design without Transfers
Elementary school choice in Boston (2012)

- Students rank any number of programs within their zone + walk-zone.

- Schools priorities over students:
  1. Continuing
  2. Siblings
  3. Walk-zone (applies to only 50% of seats)
  4. Lottery number

Main entry grade K2:
- 77 schools
- 123 programs
- ≈20-60 seats/program
Assigning students to schools

Student-proposing Deferred Acceptance (Gale-Shapley 62):

While no more students apply

- Each unmatched student applies to the next school on her list.
- Any school that has more proposals than capacity rejects its least preferred applicants

In Boston preferences of schools over students are determined by priorities and lottery numbers
Criticisms of choice plan

Selection process starts with choices, ends with luck
In towns across America, families buy homes knowing exactly where their kids will go to school. Like their post office, their parish, and their neighborhood pub, it's usually the closest one.

The high price of school assignment
By Jenna Russell and Stephanie Ebbert
Globe Staff / June 12, 2011

Like an army of yellow ants, they march across the city: 691 school buses carrying 32,221 students.

They will cost the Boston public schools a staggering $80 million next year, approaching 10 percent of the total school budget.

A daily diaspora, a scattered street
Every morning, children in Boston disperse to schools all over. Childhood chums, and neighborhood feeling, can be left behind

By Stephanie Ebbert and Jenna Russell
Globe Staff / June 12, 2011

Unpredictable
Unsustainable
Transportation Costs
Scatters Neighborhoods
Fundamental tradeoffs

- **Limit busing.**

- **Efficiency**: Match families with what is best for them.

- **Equity**: Families have reasonable chances regardless of home location or socio-economic status.

Other considerations:

- Predictability
- Simplicity
- Community cohesion
Outline

- A generalized model, applicable in more settings.
- Characterizations of “good” mechanisms in large markets
  - Characterizations for cardinal and ordinal mechanisms
- Apply the theory to Boston school choice (from large to finite)
Abstracting key issue from School Choice Example

- Limited resources.
- Private information.
- Balancing efficiency, equity, and system costs.

- Normally use auctions or queues. But money or costly signals cannot be used here.

- Other examples:
  - Course allocation.
  - Lotteries for on-campus housing.
  - Internal allocations of tasks in a company.
Large market model

- Finite set $T$ of agents types.
- Mass $n_t$ of agents of type $t \in T$.
- Finite set $S$ of services. Service $s \in S$ has capacity $m_s$.
- Each agent must be assigned 1 service.

Social planner needs to allocate services to optimize a public objective without the ability to differentiate agents via requiring monetary payments or costly effort.
Two types of mechanisms

- Cardinal (unrestricted)
- Ordinal:
  - Can only elicit preference rankings, but not intensities. *i.e.* service \(a > c > b\).
Related Work

- **School choice mechanisms:** Abdulkadiroğlu, Sönmez 03; Abdulkadiroğlu, Pathak, Roth, Sönmez 06; Abdulkadiroğlu, Pathak, Roth 09; Dur, Kominers, Pathak, Sönmez 13.
- **Assignment mechanisms:** Hylland, Zeckhauser 79, Bogomolnaia and Moulin 01, Budish 11, Miralles 12, Ashlagi, Shi 13
- **Allocation in large markets:** Thomson, Zhou 93, Zhou 92
- **Lotteries in school choice:** Pathak, Sethuraman 11; Erdil, Ergin 08.
- **Large market analysis of matching markets:** Abdulkadiroğlu, Che, Yasuda 08; Che, Kojima 11; Liu, Pycia 12; Azevedo, Leshno 10; Budish, Cantillon 12, Ashlagi, Kanoria, Leshno 13, Lee 12
- **Implementing assignment probabilities:** Budish, Che, Kojima, Milgrom 12.
- **Diversity:** Konimers, Sönmez 12; Erdil, Kumano 12; Echenique, Yenmez 13.
- **Burning mechanisms:** Hartline, Roughgarden 08, Chakravarty 10
Cardinal interim allocation rule

- An interim allocation rule for type $t$: maps reported utilities of an agent of type $t$ to assignment probabilities
  \[ x_t : U \rightarrow \Delta \]

- Incentive Compatibility (IC):
  \[ x_t(u) \text{ maximizes expected utility } v(u') = u \cdot x_t(u') \]

- Pareto efficient within type:
  - Does not exist $x'_t$ with same average allocation as $x_t$ but strictly Pareto improves $x_t$.
  - “Valid”: both IC and Pareto efficient within type.
Problem formulation

- Given utility priors $F_t$ for each type.
- Max $W(x)$
  - Can encompass social planner’s balancing of welfare, equity, system costs, and distributional preferences.
- Subject to
  - $x_t$ valid (incentive compatible and Pareto efficient within type)
  - System costs or distributional constraints.
Characterization of valid allocation rules

**Theorem:** Let $F$ be continuous and with full relative support for all types. Every valid allocation rule $x$ can be supported as a Competitive Equilibrium from Equal Incomes (CEEI) with some price vector $a \in (0, \infty)^{|S|}$.

**Interpretation:** each agent has one unit of budget and can purchase any probabilistic assignment that does not exceeds the budget.

**Optimization** – only variables are virtual prices
Proof idea

Step 1: An allocation rule is incentive compatible if and only if there exists a closed convex set $X$ such that $x(u) \in \arg\max_{y \in X} u \cdot y$
Proof idea

Step 2: Let $X$ be the corresponding convex set for the allocation rule $x$. Enough to show that there exists a unique supporting hyperplane to $X$ that intersects the interior of $\Delta$. 

![Diagram showing a triangle with vertices labeled (0,1,0), (1,0,0), and (0,0,1) and a line intersecting the interior of the triangle.](image)
Proof idea

Step 2: Let $X$ be the corresponding convex set for the allocation rule $x$. Enough to show that there exists a unique supporting hyperplane to $X$ that intersects the interior of $\Delta$.

More than one supporting hyperplane contradicts Pareto-efficiency.
Ordinal interim allocation rule

- Ordinal interim allocation rule for type $t$: maps ranking report of an agent of type $t$ to assignment probabilities

$$x_t : \Pi \rightarrow \Delta$$

- Incentive Compatibility (IC):
  - $x_t(\pi)$ maximizes expected utility $v(\pi') = u \cdot x_t(\pi')$

- Ordinal efficient within type:
  - Does not exist $x_t'$ with same average allocation as $x_t$ but strictly Pareto improves in terms of first order stochastic dominance.

- “Valid”: both IC and Ordinal efficient within type.
Characterization of valid allocation rules

Definition: Ordinal interim allocation rule $x: \Pi \to \Delta$ is **lottery-plus-cutoff** if there exists cutoffs $a_s \in [0,1]$ such that

$$x_{\pi(k)}(\pi) = \max_{j=1}^k a_{\pi(j)} - \max_{j=1}^{k-1} a_{\pi(j)}.$$

**Interpretation:** agents have lottery numbers distributed Uniform(0,1).
Can choose services $s$ for which they do not exceed the cutoff $a_s$.

**Theorem:**
Every valid ordinal interim allocation rule is lottery-plus-cutoff.
Proof sketch

Lemma: An allocation rule $x(\pi)$ is incentive compatible if and only if there exists a monotone submodular function $f: 2^{|S|} \to [0,1]$ s.t.

$$x_{\pi(k)}(\pi) = f(\{\pi(1), ..., \pi(k)\}) - f(\{\pi(1), ..., \pi(k - 1)\})$$
Lemma idea

If \( x \) is incentive compatible, then for any \( M \subseteq S \) one can define

\[
f(M) = \sum_{j=1}^{|M|} \pi_{\pi(j)}(\pi), \quad \text{where } \{\pi(1), \ldots, \pi(|M|)\} = M
\]

\( f \) is monotone and submodular
Lemma idea (cont.):

If \( x \) is defined by \( x_{\pi(k)} (\pi) = f(\{\pi(1), \ldots, \pi(k)\}) - f(\{\pi(1), \ldots, \pi(k - 1)\}) \) with monotone submodular \( f \), the range of \( x \) is the vertex set of the base polytope of the polymatroid:

\[
\sum_{s \in M} x_s \leq f(|M|) \quad \forall M \subseteq S
\]

\[
\sum_{s \in M} x_s = 1
\]

\[
x \geq 0
\]

\[\Rightarrow\] Greedy optimization with objective \( u \cdot x \) (assuming \( u_1 \geq u_2 \geq \cdots \geq u_{|S|} \)) leads to set \( x \)'s as we defined.

\[\Rightarrow\] thus \( x \) is incentive compatible
Exchange lemma:

Let $f$ be a monotone submodular function corresponding to an incentive compatible allocation rule $x$. If $x$ is Pareto efficient, then for every $M_1, M_2 \subseteq S$

$$f(M_1 \cup M_2) = \max\{f(M_1), f(M_2)\}$$
Insights from ordinal mechanisms

- A valid mechanism is equivalent to assign a menu with all services with larger cutoffs than the lottery number: *randomized menu with nested menus*
- Only variables in the optimization problem are cutoffs
Solving the optimization problem

- Suppose public objective is a **linear combination of utilitarian and max-min welfare** (or any other linear objective)

  \[ W = \alpha \sum_t w_t v_t + (1 - \alpha) \min_t v_t \]

  - **utilitarian welfare**
  - **max-min welfare**
  - **parameter**
  - **arbitrary weights**
  - **expected utility of type t**

- **Linear costs**: vector of cost \( c_{ts} \) for allocating an agent of type \( t \) to service \( s \); budget \( B \) on expected costs. \( \sum_t |n_t| x_{ts} c_{ts} \leq B \)
Optimization with randomized menus

Max $W = \alpha \sum_{t,M} w_t v_t(M) z_t(M) + (1 - \alpha) \min_t \sum_M v_t(M) z_t(M)$

s.t.

Assign a menu  $\sum_M z_t(M) = 1 \ \forall t$

Capacity  $\sum_M n_t p_t(s, M) z_t(M) \leq m_s \ \forall s$

Budget  $\sum_{s,t,M} n_t p_t(s, M) c_{ts} z_t(M) \leq C$

$z_t(M) \geq 0, \ \forall t, M \subseteq S$

Probability assigning menu $M$ to type $t$

Probability agent of type $t$ chooses service $s$ from $M$

Too many variables!
Optimization with randomized menus

Utility prior $F_t$ is “logit”: if $u_{is} = a_t s + b_t \epsilon_{is}$  $\epsilon_{is}$ i.i.d. standard Gumbel

**Theorem:** Under logit utility priors the optimal solution can be found in polynomial time.
Applying machinery to school choice

- Solve the optimal mechanism for the large market
- Translate the cutoffs from the opt mechanism to a finite market mechanism:
  - construct priorities for each school and run Deferred Acceptance
- Under same budget for total miles bused, computed optimal for:
  - $\alpha = 1$: utilitarian welfare
  - $\alpha = 0$: max-min welfare
  - $\alpha = 0.5$: Equal weighting of above
# Performance of opt under various $\alpha$’s

<table>
<thead>
<tr>
<th></th>
<th>$\alpha = 1$</th>
<th>$\alpha = 0.5$</th>
<th>$\alpha = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Average expected utility</strong></td>
<td>7.78</td>
<td>7.66</td>
<td>7.39</td>
</tr>
<tr>
<td><strong>Expected utility for worse off type</strong></td>
<td>2.52</td>
<td>7.39</td>
<td>7.39</td>
</tr>
</tbody>
</table>

Averages over 10,000 independent simulations
Elementary school choice in Boston (2012)

- Students rank any number of programs within their zone + walk-zone.

- Schools priorities students:
  1. Continuing
  2. Siblings
  3. Walk-zone (applies to only 50% of seats)
  4. lottery number

- Gale-Shapley’s Deferred Acceptance Algorithm:
  1. Student applies to top choice.
  2. Program accepts if space available; otherwise bump out least priority student.
  3. Remove choice from bumped student; iterate.

Main entry grade K2:
- 77 schools
- 123 programs
- ≈20-60 seats/program
Data

- For each student, (of approximately 4000 students), have
  - Home location (14 neighborhoods, 868 geocodes)
  - Ranked list of preferences:
    - 1st choice, 2nd choice, 3rd choice, ...

- For each school program, have
  - Location, test scores, demographics, program type, ...
Modeling demand

- Multinomial logit:

\[ u_{ts} = Q_s - D_{ts} + \omega \cdot W_{ts} + \beta \epsilon_{ts} \]

- Fit \( Q, \omega, \beta \) from micro choice data using MLE.

\( Q_s: 0-6.29 \) (additional utility in travelling distance)
\( \omega: 0.86 \) (additional utility for walk zone)
\( \beta: 1.88 \) (standard deviation of taste shock)
Proposed Solution

- **“Home Based A (Baseline)”**: Each family gets union of walk-zone, closest 2 top 25% schools, closest 4 top 50% schools, closest 6 top 75% schools, closest 3 “capacity schools.”

- **Logic:**
  - Offer “enough” schools at various thresholds.
  - Compensate families in “bad areas” with more choices.
  - Dynamically adapts to changes in quality.
    - Unlike rigid zone maps.
## Evaluation of plans

<table>
<thead>
<tr>
<th></th>
<th>Minimum</th>
<th>Baseline</th>
<th>Opt</th>
<th>Opt A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Miles busing per student</td>
<td>0.35</td>
<td>0.64</td>
<td>0.71</td>
<td>0.63</td>
</tr>
<tr>
<td>Average exp utility</td>
<td>6.31</td>
<td>6.95</td>
<td>7.62</td>
<td>7.49</td>
</tr>
<tr>
<td>Exp utility for worse off type</td>
<td>2.86</td>
<td>4.53</td>
<td>7.05</td>
<td>7.02</td>
</tr>
<tr>
<td>% getting top choice</td>
<td>66</td>
<td>64</td>
<td>80</td>
<td>79</td>
</tr>
<tr>
<td>% getting 3&lt;sup&gt;rd&lt;/sup&gt; choice</td>
<td>88</td>
<td>85</td>
<td>94</td>
<td>93</td>
</tr>
</tbody>
</table>

Averages over 10,000 independent simulations
Comparing Choice Menus for a Neighborhood Near “Higher Quality” Schools

Baseline

Opt A
Comparing choice menus for a Neighborhood near “Lower Quality” Schools
Optimal plan has larger catchment area for less popular schools
Summary of Insights

- **Possible to** simultaneously **achieve** high **efficiency**, **equity**, **predictability**, while staying within busing budget.

- **“Optimal plan” more aggressive** than Baseline compensating “lower quality” of choice with higher quantity.
  - Logic: families would only choose far away “lower quality” schools if they have a good reason, so offering to bus them to these schools is win/win.
Question: Can we improve community cohesion without affecting choice?

- “without affecting choice”: same application process, same choices, **same assignment probabilities**.
- “community cohesion”: conditional on being assigned, how many others from my community can I expect to be co-assigned with? Average this across students.
  - Proportional to # of same-community-pairs assigned together.

Ashlagi and Shi 2013: Improving community cohesion in school choice,
Characterization of valid allocation rules

Every valid ordinal interim allocation rule is lottery-plus-cutoff.

Flexibility:
- a. lotteries (can correlate)
- b. priorities
Maximize community cohesion

- $z_{is}$: random indicator for $i$ being assigned to $s$.
- $c(i)$: community of student $i$.

\[
\begin{align*}
\text{Max} & \quad \sum_s E\left[ \sum_{c(i) = c(i')} z_{is} z_{i's} \right] \quad \text{(community cohesion)} \\
\text{s.t.} & \quad \text{maintaining } p_{is} = E[z_{is}] \quad \text{for all } i, s \\
& \quad z \text{ is feasible random assignment.}
\end{align*}
\]

- Heuristic does well in simulation.
When can cohesion be improved?

C. Significant gain from correlated lottery

A. Cohesion already high

B. No hope for high cohesion

Between-community heterogeneity

Within-community heterogeneity
No Lottery to Correlate for Continuing Students and Siblings

<table>
<thead>
<tr>
<th></th>
<th>K1</th>
<th>K2</th>
</tr>
</thead>
<tbody>
<tr>
<td>% of students</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Continuing students</td>
<td>6%</td>
<td>47%</td>
</tr>
<tr>
<td>Non-continuing siblings</td>
<td>26%</td>
<td>12%</td>
</tr>
<tr>
<td>New families</td>
<td>68%</td>
<td>41%</td>
</tr>
<tr>
<td>% assigned top choice</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Continuing students</td>
<td>92%</td>
<td>95%</td>
</tr>
<tr>
<td>Non-continuing siblings</td>
<td>80%</td>
<td>79%</td>
</tr>
<tr>
<td>New families</td>
<td>24%</td>
<td>29%</td>
</tr>
</tbody>
</table>

So can only hope to significantly improve cohesion for new families.
Impact of Lottery Correlation

<table>
<thead>
<tr>
<th>Grade</th>
<th>Student Type</th>
<th>Baseline</th>
<th>Correlated</th>
<th>Upperbound</th>
<th>Gain in Cohesion</th>
</tr>
</thead>
<tbody>
<tr>
<td>K1</td>
<td>All</td>
<td>1.35</td>
<td>2.11</td>
<td>2.70</td>
<td>0.75</td>
</tr>
<tr>
<td></td>
<td>Continuing</td>
<td>1.30</td>
<td>1.32</td>
<td>1.38</td>
<td>0.02</td>
</tr>
<tr>
<td></td>
<td>Non-continuing siblings</td>
<td>1.35</td>
<td>1.43</td>
<td>1.56</td>
<td>0.08</td>
</tr>
<tr>
<td></td>
<td><strong>New families</strong></td>
<td>1.36</td>
<td>2.44</td>
<td>3.26</td>
<td>1.08</td>
</tr>
<tr>
<td>K2</td>
<td>All</td>
<td>2.48</td>
<td>2.89</td>
<td>3.39</td>
<td>0.42</td>
</tr>
<tr>
<td></td>
<td>Continuing</td>
<td>2.26</td>
<td>2.27</td>
<td>2.30</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>Non-continuing siblings</td>
<td>2.91</td>
<td>3.01</td>
<td>3.23</td>
<td>0.10</td>
</tr>
<tr>
<td></td>
<td><strong>New families</strong></td>
<td>2.61</td>
<td>3.58</td>
<td>4.69</td>
<td>0.97</td>
</tr>
</tbody>
</table>

For new families, **79% cohesion gain** over baseline for K1 and **34% for K2**. Increase # of neighbors by \( \approx 1 \).
Conclusion

- Incorporate prior information into assignment problems
- Characterization of ordinal and cardinal incentive compatible Pareto optimal (within type) mechanisms in large markets
- Efficient computation of the ordinal mechanism in an relevant empirical environment
- Engineering approach for implementing in finite markets
- Open question: solve the cardinal mechanism for “realistic” preferences
2018 Board Resolution (link):

THEREFORE BE IT RESOLVED: The SFUSD will initiate a process to develop a new student assignment system, focusing on elementary schools, which will be predicated on **greater predictability, transparency, accessibility to neighborhood options, equity, a strong commitment to integrated schools; and**

FURTHER BE IT RESOLVED: In developing the policy goals for a revised student assignment system, staff will consider: Access to a **high quality school**; and Access to a **diverse school**; and Access to a **school where sibling(s) attend**; and

BE IT FURTHER RESOLVED: In developing a revised student assignment policy, staff will develop recommendations that will strive to: **Serve the needs of historically underserved students**; and Facilitate access to an elementary school within a **reasonable geographic distance and accessible to transit**; and ● **Offer a predictable, transparent and accessible student assignment system.**

**Policy Goals:** Diversity (and integration), Predictability, Proximity, Equity of Access
Student Assignment in SF

- Residential segregation:
  - Redlining
  - 1971: Horseshoe
  - 1982-2002 Open Enrollment
Need for a New Student Assignment System in San Francisco

**Problem:**
Assigning students to public schools in San Francisco Unified School District (SFUSD)

**Choice:** Disentangle neighborhood and school segregation

**Dec 2018:** SFUSD Board of Education initiated a redesign of elementary student assignment

**Goals:** Diversity, Predictability, Proximity, Equity of Access

Image source: http://racialdotmap.demographics.coopercenter.org/
School Choice in Practice

- Deferred Acceptance (DA): NYC, Boston, Washington D.C., Denver, Seattle...
- Top Trading Cycles/ DA: San Francisco, New Orleans

Since 2012 Home-Based Plan [Shi 12]
School Choice in San Francisco: 2002-current

Elementary schools
~5,000 students, ~70 programs, ~50 schools

Families rank any number of programs

Students priorities at the schools:
1. Siblings
2. CTIP
3. Neighborhood
4. lottery number

Algorithm (2002-2018):
DA (Gale Shapley) followed by “trading cycles”

2019-present:
DA (Gale Shapley)
SFUSD Student Assignment: Goals and Challenges

Dec 2018: SFUSD Board of Education initiated a redesign of elementary school student assignment

Goals: Predictability, Proximity, Diversity, Equity of Access

Challenges
- SF residential segregation patterns (ethnic and SES)
- Many programs and types of programs
- Opt out to private/charter schools

Image source: http://racialdotmap.demographics.coopercenter.org/
SFUSD Student Assignment: Policies in Practice

Idea 1: Neighborhood Assignment
- Students attend neighborhood school
- **Problem:** Racial + socioeconomic segregation of attending schools

| Image source: | http://racialdotmap.demographics.coopercenter.org/ |

Idea 2: District-Wide Choice
- Students choose any schools, run DA or TTC
- **Problems:** Unpredictable and opaque, strategic issues, did not help with diversity

- Schools with highest % students with socio-economic need (eligible for free or reduced-price lunch)
- Schools with highest % historically underserved minorities (AALPI)
## District Policy Concepts

<table>
<thead>
<tr>
<th>Concept</th>
<th>Concept #1: Initial Assignment + Choice</th>
<th>Concept #2: Choice in Small Zones</th>
<th>Concept #3: Choice in Medium Zones</th>
</tr>
</thead>
<tbody>
<tr>
<td>Concept</td>
<td><img src="image1.png" alt="Concept" /></td>
<td><img src="image2.png" alt="Concept" /></td>
<td><img src="image3.png" alt="Concept" /></td>
</tr>
<tr>
<td><strong>Geographic Constraints</strong></td>
<td>Attendance Areas (1 school)</td>
<td>Zones (3 - 5 schools)</td>
<td>Zones (8-12 schools)</td>
</tr>
<tr>
<td><strong>Student Assignment</strong></td>
<td>Automatic assignment, then optional choice</td>
<td>Choice</td>
<td>Choice</td>
</tr>
<tr>
<td><strong>Goals</strong></td>
<td>Predictability, Proximity, Diversity</td>
<td>Predictability, Diversity, Proximity</td>
<td>Diversity, Predictability, Proximity</td>
</tr>
</tbody>
</table>
District Policy Concepts: Community Feedback

Community engagement meetings in Fall 2020
- Having *some* choice was important to most families, particularly AALPI and low-income families
- Families will find it easier to give feedback after having **specific zone boundaries**

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<td><img src="image3" alt="Concept Image" /></td>
</tr>
</tbody>
</table>

General skepticism from AALPI and low-income families
- Popular only with high-income families & families in west SF
- Popular amongst almost every demographic group
- Unpopular due to concerns about feasibility and replicating district problems in each zone
List Lengths

Fraction of students with list of given length

List length
Discussion